MATH2100 Problem Set 10 (= MATH2011 Problem Set 5)

1*: A long cylinder of material with a square cross-section occupies the region $0 < x < a$, $0 < y < a$ in space. The faces at $x = 0$, $y = 0$ and $y = a$ are held at temperature $u = 0$, and the face at $x = a$ is held at temperature $u = u_0$ (const.) until a steady temperature $u(x, y)$ is reached in the cylinder. Show that

$$u(x,y) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin[(2m-1)\pi y/a]\sinh[(2m-1)\pi x/a]}{(2m-1)\sinh[(2m-1)\pi]},$$

for $0 < x < a$, $0 < y < a$. Hence deduce that on the central axis of the cylinder,

$$u(a/2, a/2) = \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)\cosh[(2m-1)\pi/2]}.$$

(*Hint:* Use $\sinh(2z) = 2\sinh(z)\cosh(z)$.)
By considering the steady-state problem when all four faces of the cylinder are held at temperature u_0, deduce that it must also be true that $u(a/2, a/2) = u_0/4$, and hence that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)\cosh[(2m-1)\pi/2]} = \frac{\pi}{8}.$$

2*: Show that $u(x, y) = x^2 - y^2 + e^{[x^2-(y+1)^2]}\cos[2x(y + 1)]$ is a harmonic function, and find a conjugate harmonic function $v(x, y)$.
Check that $\vec{\nabla} u \cdot \vec{\nabla} v = 0$ holds for all x and y.

Practice problems: K Set 12.5 p. 562 Nos. 28, 32.

Solution to the two starred problems to be handed in by 5pm on Monday, October 22 in appropriate box on Level 3, Mathematics Bldg 67.

NB: Use a cover sheet! (Download from www.maths.uq.edu.au/courses/MATH2100 /Tutorials/cover_sheet.pdf)