INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Second Semester Examination, November, 2001

# MATH2100

## APPLIED MATHEMATICAL ANALYSIS

(Unit Courses, Inf. Tech.)

**Time: TWO Hours** for working

Ten minutes for perusal before examination begins

**Check that this examination paper has 28 printed pages!**

**CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!**

Answer as many questions as time permits, and make sure to answer AT LEAST ONE of Questions A1, A2, A3 and AT LEAST ONE of Questions B1, B2, B3.
It is expected that AT LEAST THREE questions IN TOTAL will be attempted.
All questions carry the same number of marks.
Pocket calculators without ASCII capabilities may be used.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ______________________

**EXAMINER'S USE ONLY**

| QUESTION | MARK | QUESTION | MARK |
|---|---|---|---|
| A1 | | B1 | |
| A2 | | B2 | |
| A3 | | B3 | |
| TOTAL MARKS | | | |

## PART A

Q A1. (a) Find the general solution of the system

$$\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)\,, \quad A = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

Question A1 continued on next page. 

Q A1. (a) **Working space only**

Question A1 continued on next page. 

Q A1. (b) Show that there are solutions that correspond to trajectories along the straight lines $y_2 = y_1$ and $y_2 = -y_1$ in the $y_1 y_2$-phase plane.

(c) Identify the type and stability of the equilibrium (critical) point at the origin.

Q A1. (d) Determine the slopes of trajectories where they cross the lines $y_2 = -\frac{1}{3}y_1$, $y_2 = -3y_1$, $y_2 = 0$ and $y_1 = 0$. Determine the directions of trajectories where they cross the lines $y_2 = 0$ and $y_1 = 0$. Use these results and those of Part (b) to help sketch some trajectories.

Question A1 continued next page. 

Q A1. (d) **Working space only**

Q A2. (a) Consider the system

$$y_1'(t) = y_1(t) + y_2(t)^2$$
$$y_2'(t) = y_2(t) + y_1(t)^2 \,.$$

Find all equilibrium (critical) points of the system in the $y_1 y_2$ phase-plane.

Question A2 continued on next page. 

Q A2. (b) Determine the type and stability of the critical points by linearization.

Question A2 continued on next page. 

Q A2.

(c) Show very roughly the shape of trajectories in the phase-plane near the critical points, with appropriate directions.

Q A2.

(d) Check that $y_1(t) = y_2(t) = -e^t/(1+e^t)$ is an exact solution of the nonlinear system of ODEs, and show the corresponding trajectory in the phase-plane for $-\infty < t < \infty$.

Q A3. (a) Find the Inverse Laplace Transform of $1/((s+1)(s+2))$ in two ways:

(i) using Partial Fractions

(ii) using the Convolution Theorem (see Table) with $F(s) = 1/(s+1)$ and $G(s) = 1/(s+2)$.

Question A3 continued on next page. 

Q A3.

(b) Use the First Shifting Theorem (see Table) to show that

$$\mathcal{L}((t+2)^2 e^{3t}) = \frac{4s^2 - 20s + 26}{(s-3)^3}$$

Q A3. (c) Use the Method of Laplace Transforms to solve

$$y''(t) + 5y'(t) + 6y(t) = 2u(t-1)\,, \qquad y(0) = 0\,, \qquad y'(0) = 1\,,$$

where $u(t)$ is Heaviside's Step Function (see Table).

Question A3 continued next page. 

Q A3. (c) **Working space only**

## PART B

Q B1. (a) Show using integration by parts that

$$\int x\cos(px)\,dx = \frac{x}{p}\sin(px) + \frac{1}{p^2}\cos(px) + \text{const.}\,,$$

$$\int x\sin(px)\,dx = -\frac{x}{p}\cos(px) + \frac{1}{p^2}\sin(px) + \text{const.}\,.$$

Question B1 continued on next page. 

Q B1. (b) Show that the Fourier Series corresponding to the function defined by

$$f(x) = \alpha x/L\,, \qquad -L < x < 0\,; \qquad f(x) = \beta x/L\,, \qquad 0 < x < L\,;$$
$$\text{and} \qquad f(x+2L) = f(x)\,, \qquad -\infty < x < \infty\,,$$

where $\alpha$ and $\beta$ are constants, is

$$(\alpha - \beta)\left(-\frac{1}{4} + \frac{2}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos[(2m+1)\pi x/L]}{(2m+1)^2}\right) - \frac{(\alpha+\beta)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin[n\pi x/L]}{n}\,.$$

Question B1 continued on next page. 

Q B1. (b) **Working space only**

Question B1 continued on next page. 

Q B1. (c) Explain what happens when (i) $\alpha = \beta$ (ii) $\alpha = -\beta$.

(d) By considering the values to which the series should converge at $x = L$ and $x = L/2$, deduce that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}, \qquad \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

Q B1. (d) **Working space only**

Question B2 see next page.

Q B2. (a) You are given (**no need to check!**) that the function $G(x-y,t)$ defined by

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\; e^{-(x-y)^2/(4c^2t)}$$

satisfies the 1-dimensional Heat Equation:

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

Show that $u(x,t)$ defined by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\,dy$$

satisfies the 1-dimensional Heat Equation for $-\infty < x < \infty$ and $t > 0$, and also the initial condition

$$\lim_{t\to 0_+} u(x,t) = f(x)\,, \qquad -\infty < x < \infty\,.$$

Question B2 continued on next page. 

Q B2.

(b) Show that in the case when $f(x) = u_0$ (const.) for $x > 0$ and $f(x) = u_1$ (const.) for $x < 0$, this gives

$$u(x,t) = \frac{1}{2}(u_0 + u_1) + \frac{1}{2}(u_0 - u_1)\ \mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right),$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv\,.$$

Question B2 continued on next page. 

Q B2. (b) **Working space only**

Question B3 see next page. 

Q B3. (a) A rod of iron of square cross-section occupies the region $0 < x < a$, $0 < y < a$, and $-\infty < z < \infty$. The faces at $x = 0$ and $x = a$ are thermally insulated, while the face at $y = 0$ is maintained at temperature $u = 0$. On the face at $y = a$, the steady temperature $u(x, a) = f(x)$ is maintained, with $f(x)$ a given function for $0 < x < a$. After a long time, the rod reaches a steady temperature distribution $u(x, y)$, for $0 < x < a$, $0 < y < a$. What PDE and BCs determine the form of $u(x, y)$?

(b) Work carefully through Fourier's Method of Separation of Variables and Superposition to obtain the solution

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{a}) \sinh(\frac{n\pi y}{a})$$

where

$$A_0 = \frac{1}{a^2} \int_0^a f(x)\, dx$$

$$A_n = \frac{2}{a \sinh(n\pi)} \int_0^a f(x) \cos(\frac{n\pi x}{a})\, dx\,.$$

Question B3 continued on next page. 

Q B3. (b) **Working space only**

Q B3. (b) **Working space only**

Question B3 continued on next page. 

Q B3. (b) **Working space only**

Q B3. (c) Evaluate $u(x, y)$ in the case that $f(x) = u_0$ (const.). Can you see from the symmetry of the situation why the solution is so simple in this case?

Table of Laplace Transforms see next page. 

Table of Laplace Transforms see next page. 