INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Second Semester Examination, November, 2002

MATH2100

APPLIED MATHEMATICAL ANALYSIS

(Unit Courses, Inf. Tech.)

Time: TWO Hours for working

Ten minutes for perusal before examination begins

Check that this examination paper has 24 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!

Answer as many questions as time permits, and make sure to answer AT LEAST ONE of Questions A1, A2, A3 and AT LEAST ONE of Questions B1, B2, B3.
It is expected that AT LEAST THREE questions IN TOTAL will be attempted.
All questions carry the same number of marks.
Pocket calculators without ASCII capabilities may be used.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
A1		B1	
A2		B2	
A3		B3	
TOTAL MARKS			

PART A

QA1. (a) Find the general solution of the system

$$\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)\,, \quad A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Question A1(a) continued on next page.

QA1. (a) **Working space only**

Question A1 continued on next page.

QA1. (b) Write the general solution in real form.

(c) Identify the type and stability of the equilibrium (critical) point at the origin.

Question A1 continued on next page.

QA1. (d) Determine the slopes of trajectories where they cross the lines $y_2 = y_1$, $y_2 = -y_1$, $y_2 = 0$ and $y_1 = 0$. Determine the directions of trajectories where they cross the line $y_1 = 0$. Use your results to help sketch some trajectories.

Question A2 see next page.

QA2. (a) Rewrite

$$y''(t) + 4y(t) - y(t)^3 = 0$$

as a system of two coupled 1st-order ODEs for $y_1(t) = y(t)$, $y_2(t) = y'(t)$, and locate the equilibrium points of the system in the (y_1, y_2)-phase plane.

Question A2 continued on next page.

QA2. (b) Linearize the system near each of the equilibrium points in turn, and determine the type and stability of the equilibrium point in each case.

Question A2 continued on next page.

QA2. (c) Returning to the system in Part (a), write down the 1st-order ODE satisfied by $y_2(y_1)$, that is to say, y_2 regarded as a function of y_1.

(d) Check that $y_2 = \pm(1/\sqrt{2})(y_1^2 - 4)$ is a solution of the ODE in (c). Sketch these two curves in the (y_1, y_2) plane. Now add the critical points to your sketch, showing roughly some trajectories with directions near each.

Question A3 see next page.

QA3. (a) Using the Table provided,

(i) find the Laplace Transform of

$$f(t) = 3u(t - 3\pi)\sin t$$

where $u(t)$ is the Heaviside Step Function, and sketch the function $f(t)$.

Question A3 continued on next page.

QA3. (a) (ii) Show that the Inverse Laplace Transform of

$$F(s) = \frac{s^2 - 1}{(s^2 + 1)^2}$$

is $f(t) = t \cos t$.

Question A3 continued on next page.

QA3. (b) Use the method of Laplace Transforms to solve the equation

$$y''(t) + y(t) = -2\sin t$$

with Initial Conditions $y(0) = 3$, $y'(0) = 4$.

Question B1 see next page.

PART B

Q B1. (a) Show using integration by parts that

$$\int x^2 \cos(px)\, dx = \frac{x^2}{p}\sin(px) + \frac{2x}{p^2}\cos(px) - \frac{2}{p^3}\sin(px) + \text{const.}\,,$$

Question B1 continued on next page.

Q B1. (b) Deduce that, when $0 < x < L$,

$$x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(\frac{n\pi x}{L}) \quad (*) .$$

Question B1 continued on next page.

Q B1.

(c) Sketch the graph of the function to which the RHS of equation $(*)$ converges, for $-\infty < x < \infty$. Then deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}\,, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Question B2 see next page.

Q B2. (a) Check that the function $G(x,t)$ defined by

$$G(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\; e^{-x^2/(4c^2t)}$$

satisfies the 1-dimensional Heat Equation:

$$G_t(x,t) = c^2 G_{xx}(x,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

Question B2 continued on next page.

Q B2. (a) **Working space only**

Question B2 continued on next page.

Q B2. (b) Consider the problem of heat conduction in the top layers of earth, assumed to occupy $x > 0$, and to have thermometric conductivity $c^2 \approx 2 \times 10^{-3}\mathrm{cm}^2\mathrm{sec}^{-1}$. Yearly periodic heating of the surface by the sun is assumed to produce the boundary condition $u(0,t) = u_0 \cos(\omega t)$, where $\omega = 2\pi\ \mathrm{yr}^{-1}$. Look for a complex solution of the 1-dimensional heat equation in the form

$$u(x,t) = A(x)e^{i\omega t}\,,$$

whose real part satisfies the boundary condition. Hence obtain the real solution

$$u(x,t) = u_0 e^{-bx}\cos(\omega t - bx)\,, \quad b = \sqrt{\omega/(2c^2)}\,.$$

Question B2 continued on next page.

Q B2.

(c) Check that $bx \approx \pi/4$ and $e^{-bx} \approx 1/2$ when $x = 100$cm, and hence deduce that at a depth of about 4 metres, temperature fluctuations are reduced in magnitude by a factor of 16, and it is mid-Winter there when it is mid-Summer on the surface.

Question B3 see next page.

Q B3. (a) Show carefully that the only solutions of the form $u(x,t) = F(x)G(t)$ of the PDE

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)\,, \quad 0 < x < L\,, \quad t > 0\,,$$

and the BCs

$$u(0,t) = 0 = u(L,t)\,, \quad t > 0\,,$$

are

$$u_n(x,t) = \sin(\frac{n\pi x}{L})\big(A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L})\big)\,, \quad n = 1,\, 2,\, 3,\, \ldots$$

where A_n and B_n are arbitrary constants.

Question B3 continued on next page.

Q B3. (a) **Working space only.**

Question B3 continued on next page.

Q B3. (b) A string is stretched between $x = 0$ and $x = L$ with its ends fixed, and is subject to an applied force per unit mass $q(x) = q^* \sin(\pi x/L)$. The string is released from rest with an initial displacement $u(x, 0) = u^* \sin(2\pi x/L)$. Here q^* and u^* are constants. Deduce that

$$u(x,t) = \frac{q^* L^2}{c^2\pi^2}[1 - \cos(\pi ct/L)] \sin(\pi x/L) + u^* \cos(2\pi ct/L) \sin(2\pi x/L)\,, \quad t > 0\,.$$

Question B3 continued on next page.

Q B3. (b) **Working space only**

Q B3. (b) **Working space only**

Table of Laplace Transforms see next page.

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