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The University Of Queensland

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Second Semester Examination, November, 2003

MATH2100

APPLIED MATHEMATICAL ANALYSIS

(Unit Courses, Inf. Tech.)

Time: TWO Hours for working

Ten minutes for perusal before examination begins

Check that this examination paper has 28 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!

This paper is worth 65% of the total assessment for MATH2100. Each question is worth 13.5 marks. You can score at most 65 marks for the whole paper. Answer as many questions as time permits, and make sure to answer AT LEAST ONE of Questions A1, A2, A3 and AT LEAST ONE of Questions B1, B2, B3. Pocket calculators without ASCII capabilities may be used.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
A1		B1	
A2		B2	
A3		B3	
TOTAL MARKS			

PART A

Q A1 (a). Find the general solution of the system

$$\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)\,, \quad A = \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix}.$$

Question A1 continued on next page.

Q A1 (a). **Working space only**

Q A1 (b). Write the general solution in real form.

Question A1 continued on next page.

Q A1 (b). **Working space only**

Q A1 (c). Identify the type and stability of the equilibrium (critical) point at the origin.

Question A1 continued on next page.

Q A1 (d). Determine the slopes of trajectories where they cross the lines $y_2 = y_1$, $y_2 = -y_1$, $y_2 = -5y_1$ and $y_1 = 0$. Determine the directions of trajectories where they cross the line $y_1 = 0$. Use your results to help sketch some trajectories.

Question A1 continued next page.

Q A1 (d). **Working space only**

Q A2 (a). Rewrite

$$y''(t) + 4y(t) - 2y(t)^3 = 0$$

as a system of two coupled 1st-order ODEs for $y_1(t) = y(t)$, $y_2(t) = y'(t)$, and locate the equilibrium points of the system in the (y_1, y_2)-phase plane.

Question A2 continued on next page.

Q A2 (b). Linearize the system near each of the equilibrium points in turn, and determine the type and stability of the equilibrium point in each case.

Question A2 continued on next page.

Q A2 (c). Returning to the system in Part (a), write down the 1st-order ODE satisfied by $y_2(y_1)$, that is to say, y_2 regarded as a function of y_1.

Q A2 (d) Check that $y_2 = \pm(y_1^2 - 2)$ is a solution of the ODE in (c). Sketch these two curves in the (y_1, y_2) plane. Now add the critical points to your sketch, showing roughly some trajectories with directions near each.

Q A3 (a). Use the Convolution Theorem (see Table) to find the Inverse Laplace Transform of $H(s) = F(s)G(s)$ in the case

$$F(s) = \frac{1}{s+3} \quad , \quad G(s) = \frac{1}{s^2}.$$

Question A3 continued on next page.

Q A3 (b). Given

$$f(t) = \begin{cases} 3 & : \quad 0 < t < \pi \\ -1 & : \quad \pi < t < 3\pi \\ \cos(t) & : \quad t > 3\pi \end{cases} ,$$

sketch the graph of $f(t)$ and then find its Laplace transform.

Question A3 continued on next page.

Q A3 (c). Use the Method of Laplace Transforms to solve

$$y''(t) - 7y'(t) + 6y(t) = 3u(t-1)\,, \qquad y(0) = -1\,, \qquad y'(0) = 1\,,$$

where $u(t)$ is Heaviside's Step Function (see Table).

Question A3 continued next page.

Q A3 (c). **Working space only**

Question B1 see next page.

PART B

Q B1 (a). Use the formulas

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)]\,; \quad \sin(A)\cos(B) = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

to deduce that if $n \neq 1$, then

$$\int \sin(nx)\sin(x)\,dx = \frac{1}{2}\left(\frac{1}{n-1}\sin[(n-1)x] - \frac{1}{n+1}\sin[(n+1)x]\right)$$
$$\int \cos(nx)\sin(x)\,dx = \frac{1}{2}\left(\frac{1}{n-1}\cos[(n-1)x] - \frac{1}{n+1}\cos[(n+1)x]\right)\,,$$

and that

$$\int \sin^2(x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin[2x]\right)\,; \quad \int \cos(x)\sin(x)\,dx = -\frac{1}{4}\cos(2x)\,.$$

Question B1 continued on next page.

Q B1 (b). Show that the Fourier Series corresponding to the function defined by

$$f(x) = 0\,, \qquad -\pi < x < 0\,; \qquad f(x) = \sin(x)\,, \qquad 0 < x < \pi\,;$$
$$\text{and} \qquad f(x + 2\pi) = f(x)\,, \qquad -\infty < x < \infty\,,$$

is

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\left(\frac{1}{(1)(3)}\cos(2x) + \frac{1}{(3)(5)}\cos(4x) + \cdots\right).$$

Question B1 continued on next page.

Q B1 (b). **Working space only**

Question B1 continued on next page.

Q B1 (b). **Working space only**

Question B1 continued on next page.

Q B1 (c). By considering the value to which the series should converge at $x = \pi/4$, deduce that

$$\frac{1}{(3)(5)} - \frac{1}{(7)(9)} + \frac{1}{(11)(13)} \cdots = \frac{\pi}{4\sqrt{2}} - \frac{1}{2}\,.$$

Q B2 (a). You are given (**no need to check!**) that the function $G(x-y,t)$ defined by

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-(x-y)^2/(4c^2t)}$$

satisfies

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

Show that $u(x,t)$ defined by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

satisfies the 1-dimensional Heat Equation for $-\infty < x < \infty$ and $t > 0$, and also the initial condition

$$\lim_{t\to 0_+} u(x,t) = f(x)\,, \qquad -\infty < x < \infty\,.$$

Question B2 continued on next page.

Q B2 (b). Show that in the case when $f(x) = u_0$ (const.) for $x > 0$ and $f(x) = u_1$ (const.) for $x < 0$, this gives

$$u(x,t) = \frac{1}{2}(u_0 + u_1) + \frac{1}{2}(u_0 - u_1)\ \mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right),$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv\,.$$

Question B2 continued on next page.

Q B2 (b). **Working space only**

Question B3 see next page.

Q B3 (a). A stretched string lies between $x = 0$ and $x = L$ with its ends fixed. It is released from rest at time $t = 0$ with an initial transverse velocity that is equal to 0 for $0 < x < L/3$; equal to u_0 for $L/3 < x < 2L/3$; and equal to 0 for $2L/3 < x < L$. Assuming that small transverse vibrations occur for $t > 0$, with displacement $u(x,t)$, write down the PDE, BCs and ICs appropriate to this situation.

Q B3 (b). Find all solutions of the PDE and BCs that have the form

$$u(x,t) = F(x)G(t)\,.$$

Question B3 continued on next page.

Q B3 (b). **Working space only**

Q B3 (b). **Working space only**

Question B3 continued on next page.

Q B3 (c). Use the formulas in the second line of Q B1 (a) to check that each of your solutions can be written in the form

$$u(x,t) = F_1(x-ct) + F_2(x+ct)$$

for suitable functions F_1, F_2. What is the physical interpretation of this result?

Q B3 (d). Using your solutions obtained in (b), deduce that the displacement of the string described in (a) is given by

$$\begin{aligned} u(x,t) = \frac{2u_0L}{c\pi^2}\sum_{k=0}^{\infty} \quad & \left\{ \frac{1}{(6k+1)^2}\sin\left[\frac{(6k+1)\pi ct}{L}\right]\sin\left[\frac{(6k+1)\pi x}{L}\right]\right. \\ & + \frac{1}{(6k+5)^2}\sin\left[\frac{(6k+5)\pi ct}{L}\right]\sin\left[\frac{(6k+5)\pi x}{L}\right] \\ & \left. - \; 2\,\frac{1}{(6k+3)^2}\sin\left[\frac{(6k+3)\pi ct}{L}\right]\sin\left[\frac{(6k+3)\pi x}{L}\right]\right\}. \end{aligned}$$

Question B3 continued on next page.

Q B3 (d). **Working space only**

Question B3 continued on next page.

Table of Laplace Transforms see next page.