THE UNIVERSITY OF QUEENSLAND

St Lucia Campus

STUDENT NAME: Fred Bloggs

STUDENT NUMBER: S 1 2 3 4 5 6 7 8

SEAT NUMBER: ------------------

FINAL Examination
Second Semester, 2006

COURSE CODE	MATH2100		
COURSE TITLE	Applied Mathematical Analysis		
EXAMINER	Prof. A.J. Bracken	WEIGHTING/MARKS	65
PERUSAL TIME	10 mins No writing permitted during perusal	WRITING TIME	2 Hrs 0 mins
		NO. OF PAGES IN EXAM PAPER (*including cover sheet*)	21

Exam Type: Closed Book
Calculator: Yes - ~~Non-programmable~~ but all memory to be cleared.
Dictionary: No
Permitted Materials:

SPECIAL INSTRUCTIONS TO STUDENTS:
Answer all four questions. Marks for each part question are shown after that part. The total marks available are 65. Your score on the paper will be added to your score out of 35 for the assignments to give your final mark out of 100.

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

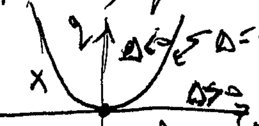
For Question 1 see next page.

Find the general solution of

Q1. (a) **Sketch the trajectories of the following system in the phase plane,** indicating the direction of flow, and **classify the type and stability** of the critical point at the origin.

$p = -6$, $q = 9-4 = 5$, $\Delta = p^2 - 4q = 16$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \dot{\underset{\sim}{y}} = A \underset{\sim}{y}$$



improper node
stable, attractive

State the equation of any straight line trajectories.
Determine the slope of the trajectories on the y_1 and y_2 axes and indicate your results on your sketch. **(8.5 marks)**

$$A = \begin{bmatrix} -3 & 1 \\ 4 & -3 \end{bmatrix}$$

E'Values: $\begin{vmatrix} -3-\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda+3)^2 - 4 = 0$

$\Rightarrow \lambda^2 + 6\lambda + 5 = 0 \Rightarrow (\lambda+5)(\lambda+1) = 0$

$\Rightarrow \lambda = -1, -5$

E'Vectors: $\lambda = -1$:

$$\begin{bmatrix} -3+1 & 1 \\ 4 & -3+1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -2u + v = 0 \\ 4u - 2v = 0 \end{matrix}\right\} \Rightarrow v = +2u$$

Take $u = 1 \Rightarrow v = -2$, so

$\underset{\sim}{x} = \begin{pmatrix} 1 \\ +2 \end{pmatrix}$

$\lambda = -5$:

$$\begin{bmatrix} -3+5 & 1 \\ 4 & -3+5 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 2u + v = 0 \\ 4u + 2v = 0 \end{matrix}\right\} \Rightarrow v = -2u$$

Take $u = 1$, so $v = -2$ & $\underset{\sim}{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Gen. Sol^n

$$\underset{\sim}{y} = A \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}$$

If $B = 0$: $y_1 = Ae^{-t}$, $y_2 = 2Ae^{-t} \Rightarrow y_2 = 2y_1$

If $A = 0$: $y_1 = Be^{-5t}$, $y_2 = -2Be^{-5t} \Rightarrow y_2 = -2y_1$

— See as $t \to \infty$, $\underset{\sim}{y} \to \underset{\sim}{0}$ in each case

Question 1 continued on next page.

Q 1 (a). **Working space only**

Slope equation: $\dfrac{dy_2}{dy_1} = \dfrac{\dot{y}_2}{\dot{y}_1} = \dfrac{4y_1 - 3y_2}{-3y_1 + y_2}$

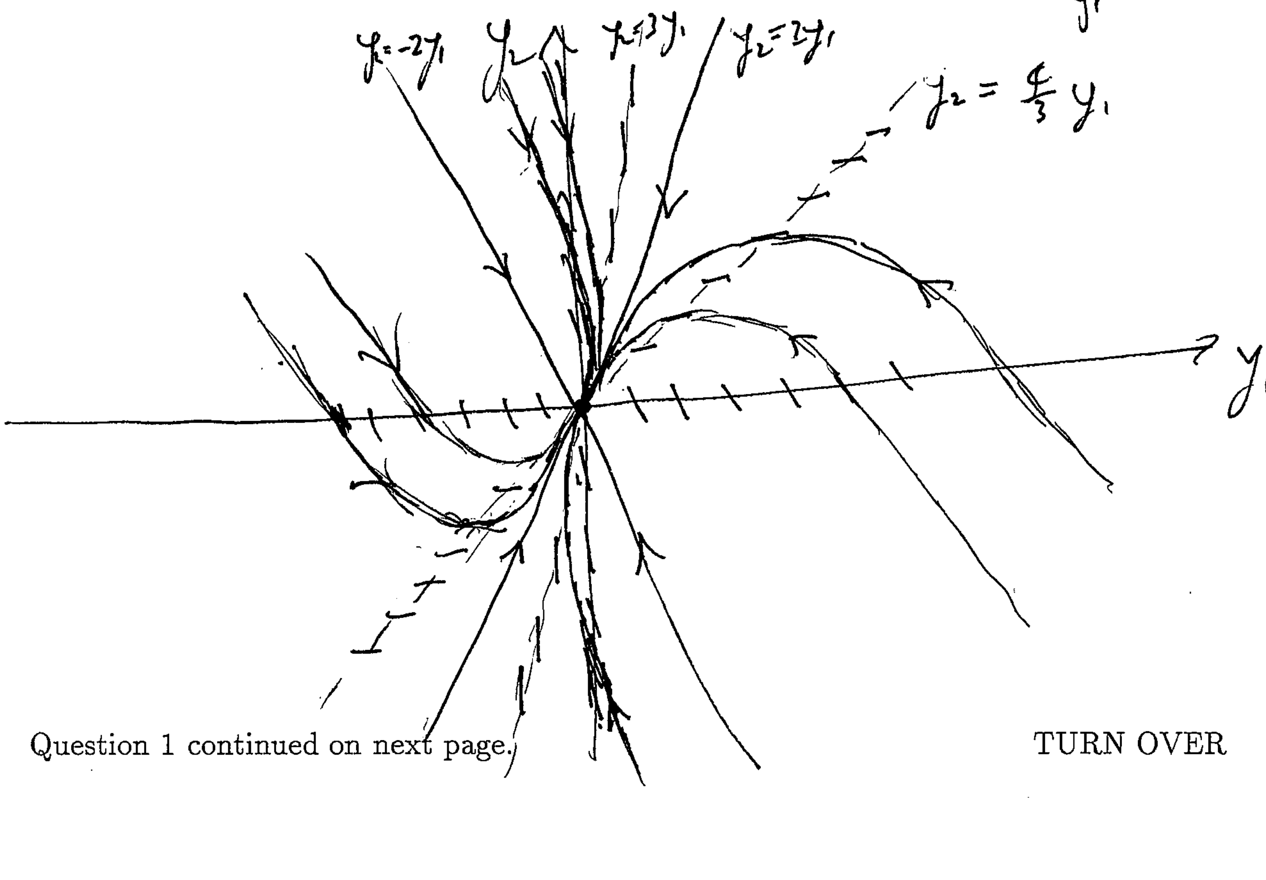
$\dfrac{dy_2}{dy_1} = 0 \Leftrightarrow 4y_1 = 3y_2$ or $y_2 = \frac{4}{3} y_1$

$\dfrac{dy_2}{dy_1} = \pm\infty \Leftrightarrow -3y_1 + y_2 = 0$ or $y_2 = 3y_1$

When $y_2 = 0$ (y_1 axis) $\dfrac{dy_2}{dy_1} = -\frac{4}{3}$

When $y_1 = 0$ (y_2 axis) $\dfrac{dy_2}{dy_1} = -3.$

Directions: When $y_2 = 0$, $\dfrac{dy_1}{dt} = -3y_1$, so y_1 incr $\Leftrightarrow y_1 < 0$
y_1 decr $\Leftrightarrow y_1 > 0$



Question 1 continued on next page.

Q1. (b) Find all the **critical points** for the following nonlinear system.

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 2r + 3rs - 4r^2 \\ 3s + rs - s^2 \end{pmatrix}$$

$y_1 \equiv r$
$y_2 \equiv s$

Then use Linearization to find the type and stability of the critical points **that do not** lie on the r axis, i.e. for which $s \neq 0$.

(9 marks)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} F_1(y_1,y_2) \\ F_2(y_1,y_2) \end{pmatrix} \qquad \begin{aligned} F_1(y_1,y_2) &= 2y_1 + 3y_1y_2 - 4y_1^2 \\ F_2(y_1,y_2) &= 3y_2 + y_1y_2 - y_2^2 \end{aligned}$$

Critical points: $F_1 = 0$ and $F_2 = 0$

$y_1(2+3y_2-4y_1) = 0$ and $y_2(3+y_1-y_2) = 0$

For critical points with $y_2 \neq 0$, $3 + y_1 - y_2 = 0$

i.e. $y_2 = y_1 + 3$

and either $y_1 = 0 \Rightarrow y_2 = 3$

or $2 + 3y_2 - 4y_1 = 0$

$\Rightarrow 2 + 3(y_1+3) - 4y_1 = 0$

$\Rightarrow y_1 = 11 \Rightarrow y_2 = 14$

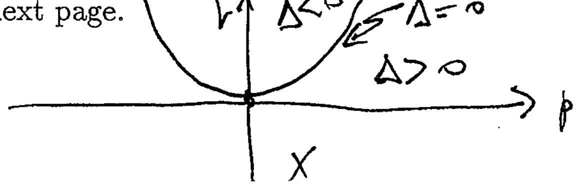
So critical points are $(0,3)$ and $(11,14)$.

$$F = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 2+3y_2-8y_1 & 3y_1 \\ y_2 & 3+y_1-2y_2 \end{bmatrix}$$

At $(0,3)$, $\dot{\underset{\sim}{Y}} \approx F\underset{\sim}{Y}$ where $\underset{\sim}{Y} = \underset{\sim}{y} - \binom{0}{3}$

& $F = \begin{bmatrix} 11 & 0 \\ 3 & -3 \end{bmatrix} \Rightarrow p = 8,\ q = -33,\ \Delta = 64 + 132 = 196$

Question 1 continued on next page.



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Saddle pt.
unstable

Q 1 (b). **Working space only**

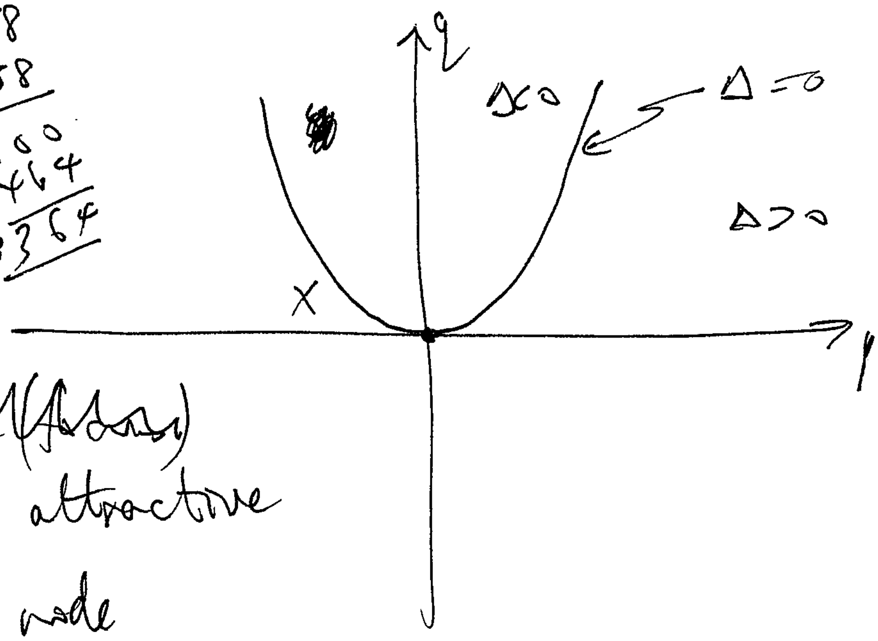
At $(11,14)$, $\dot{\underset{\sim}{Y}} \approx F\underset{\sim}{Y}$ where $\underset{\sim}{Y} = \underline{y} - \begin{pmatrix} 11 \\ 14 \end{pmatrix}$

& $F = \begin{bmatrix} 44-88 & 33 \\ 14 & 14-28 \end{bmatrix} = \begin{bmatrix} -44 & 33 \\ 14 & -14 \end{bmatrix}$

$\Rightarrow p = -58$, $q = (44)(14) - (14)(33) = (11)(14) = 154$

$\Delta = p^2 - 4q = (58)^2 - 616 > 0$

58
58
2900
464
3364



Stable & attractive

Improper node

Q2. (a) If $G(s)$ is the Laplace Transform of $g(t)$, prove carefully that

$$\frac{dG(s)}{ds} \quad \text{is the Laplace Transform of} \quad -tg(t).$$

Hence, or otherwise, determine the inverse Laplace Transform of

$$F(s) = \frac{2s}{(s^2+1)^2}.$$

(6 marks)

$\cancel{\mathcal{L}(-tg(t)) = \int_0^\infty -t g(t) e^{-st}\, dt}$

$$G(s) = \int_0^t e^{-st} g(t)\, dt$$

$$\Rightarrow \quad \frac{dG(s)}{ds} = \int_0^t \frac{d}{ds}\left(e^{-st}\right) g(t)\, dt = \int_0^t -t e^{-st} g(t)\, dt$$

$$= \mathcal{L}\left(-t\, g(t)\right).$$

$$F(s) = \frac{2s}{(s^2+1)^2} = \frac{d}{ds}\left[\frac{-1}{(s^2+1)}\right]$$

Now $\dfrac{-1}{s^2+1} = \mathcal{L}(+\sin(-t))$

So $\dfrac{2s}{(s^2+1)^2} = \mathcal{L}\left[-t \sin(-t)\right)$

$$= \mathcal{L}(t \sin t)$$

Q2. (b) Use Laplace Transforms to solve the following initial value problem.

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t) = \left\{ \begin{array}{cc} 0 & t < 3 \\ e^{-2t} & 3 \le t \end{array} \right] \quad \text{with } y(0) = 0 \text{ and } \frac{dy}{dt}(0) = 0$$

(9 marks)

Let $Y(s) = \mathcal{L}(y(t))$

Then $\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$

$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

Also $\mathcal{L}(f(t)) = \mathcal{L}(e^{-2t}\mathcal{U}(t-3))$

$= e^{-6}\mathcal{L}(e^{-2(t-3)}u(t-3))$

$= e^{-6} \cdot \dfrac{e^{-3s}}{s+2}$

So

$s^2Y + 5sY + 6Y = e^{-6}\dfrac{e^{-3s}}{s+2}$

$(s+3)(s+2)Y =$

$Y = e^{-6}e^{-3s}\dfrac{1}{(s+3)(s+2)^2}$

Now

$\dfrac{1}{(s+3)(s+2)^2} = \dfrac{A}{s+3} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$

$1 = A(s+2)^2 + B(s+3)(s+2) + C(s+3)$

$s = -2 \Rightarrow 1 = 0 + 0 + C \Rightarrow C = 1$

$s = -3 \Rightarrow 1 = A + 0 + 0 \Rightarrow A = 1$

Coeff of s^2: $0 = A + B \Rightarrow B = -A = -1$

So $Y = e^{-6}e^{-3s}\left[\dfrac{1}{s+3} - \dfrac{1}{s+2} + \dfrac{1}{(s+2)^2}\right]$

Question 2 continued on next page.

Q 2 (b). **Working space only**

Then $y(t) = e^{-6}\left[e^{-3(t-3)} - e^{-2(t-3)} + (t-3)e^{-2(t-3)}\right]u(t-3)$

$= \left[e^{3}e^{-3t} - e^{-2t} + (t-3)e^{-2t}\right]u(t-3)$

$= \begin{cases} e^{3}e^{-3t} + e^{-2t}(t-4) & t \geq 3 \\ 0 & t < 3 \end{cases}$

Question 3 on next page.

Q 3 (a). Use integration by parts to show that if p is a nonzero constant, then

$$\int x\cos(px)\,dx = \frac{x}{p}\sin(px) + \frac{1}{p^2}\cos(px) + \text{const.},$$
$$\int x\sin(px)\,dx = -\frac{x}{p}\cos(px) + \frac{1}{p^2}\sin(px) + \text{const.}.$$

(2.5 marks)

$$\int \underset{u}{x}\,\underset{v'}{\cos(px)}\,dx = \frac{x\sin(px)}{p} - \frac{1}{p}\int 1\sin(px)\,dx$$
$$= \frac{x\sin(px)}{p} + \frac{1}{p^2}\cos(px) + c$$

$$\int x\sin(px)\,dx = -x\frac{\cos(px)}{p} + \frac{1}{p}\int \cos(px)\,dx$$
$$= -x\frac{\cos(px)}{p} + \frac{1}{p^2}\sin(px) + c$$

Question 3 continued on next page.

Q 3 (b). Show that the Fourier Series corresponding to the function defined by

$$f(x) = A(1+x)\,, \qquad -1 \le x \le 0\,; \qquad f(x) = B(1-x)\,, \qquad 0 < x < 1\,;$$
$$\text{and} \qquad f(x+2) = f(x)\,, \qquad -\infty < x < \infty\,,$$

$p=2,\ L=1$

where A and B are constants, is given by

$$\frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos([2m+1]\pi x) - \frac{(A-B)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)\,. \qquad (*)$$

Series is $a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$

(8.5 marks)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \tfrac{1}{2}A\int_{-1}^{0}(1+x)\,dx + \tfrac{1}{2}B\int_{0}^{1}(1-x)\,dx$$
$$= \tfrac{1}{2}A\left[x + \tfrac{1}{2}x^2\right]_{-1}^{0} + \tfrac{1}{2}B\left[x - \tfrac{1}{2}x^2\right]_{0}^{1}$$
$$= \tfrac{1}{2}A\left[0 - (-1+\tfrac{1}{2})\right] + \tfrac{1}{2}B\left[(1-\tfrac{1}{2}) - 0\right]$$
$$= \underline{\tfrac{1}{4}(A+B)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = A\int_{-1}^{0}(1+x)\cos(n\pi x)\,dx + B\int_{0}^{1}(1-x)\cos(n\pi x)\,dx$$
$$= A\left\{\cancel{\left[\frac{1+x}{n\pi}\sin(n\pi x)\right]_{-1}^{0}} - \frac{1}{n\pi}\int_{-1}^{0}\sin(n\pi x)\,dx\right\}$$
$$+ B\left\{\cancel{\left[\frac{1-x}{n\pi}\sin(n\pi x)\right]_{0}^{1}} + \frac{1}{n\pi}\int_{0}^{1}\sin(n\pi x)\,dx\right\}$$
$$= +\frac{A}{(n\pi)^2}\left[\cos(n\pi x)\right]_{-1}^{0} - \frac{B}{(n\pi)^2}\left[\cos(n\pi x)\right]_{0}^{1}$$
$$= +\frac{A}{(n\pi)^2}\left[1-(-1)^n\right] - \frac{B}{(n\pi)^2}\left[(-1)^n - 1\right]$$
$$= \frac{A+B}{(n\pi)^2}\left[1-(-1)^n\right] = \begin{cases} \dfrac{2(A+B)}{((2m+1)\pi)^2} & n = 2m+1 \text{ odd},\ m = 0,1,2,\ldots \\ 0 & n = 2m \text{ even},\ m = 1,2,\ldots \end{cases}$$

Question 3 continued on next page.

Q 3 (b). **Working space**

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x)\,dx = A\int_{-1}^{0}(1+x)\sin(n\pi x)\,dx + B\int_{0}^{1}(1-x)\sin(n\pi x)\,dx$$

$$= A\left\{\left[-\frac{(1+x)}{n\pi}\cos(n\pi x)\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos(n\pi x)\,dx\right\}$$

$$+ B\left\{\left[-\frac{(1-x)}{n\pi}\cos(n\pi x)\right]_{0}^{1} - \frac{1}{n\pi}\int_{0}^{1}\cos(n\pi x)\,dx\right\}$$

$$= \frac{A}{n\pi}\left[-1-0\right] + \cancel{\frac{A}{n^2\pi^2}\left[\sin(n\pi x)\right]_{-1}^{0}}$$

$$+ \frac{B}{n\pi}\left[0+1\right] - \cancel{\frac{B}{n^2\pi^2}\left[\sin(n\pi x)\right]_{0}^{1}}$$

$$= -\frac{(A-B)}{n\pi}.$$

$\mathcal{F}$ series is

$$\frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos\left[(2m+1)\pi x\right]$$

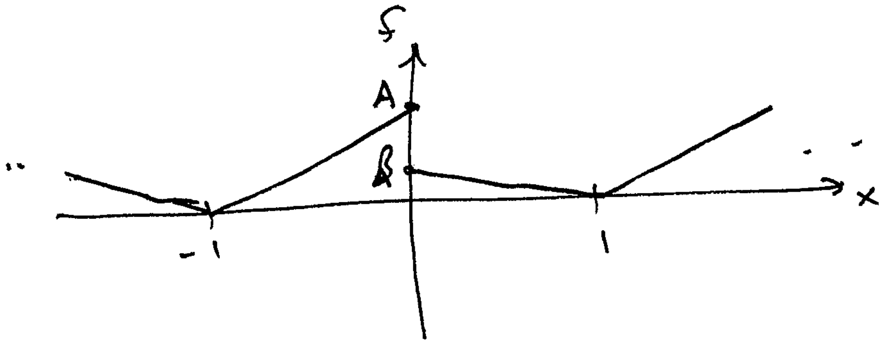
$$-\frac{(A-B)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x).$$

Question 3 continued on next page.

Q 3 (c). To what value must the series (∗) converge at $x = 0$? Use this result to deduce that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

(2 marks)



f is piecewise conts ~~with~~ and $f(0_-)$, $f(0_+)$, $f'(0_-)$, $f'(0_+)$ well-defined.

∴ series converges to $\frac{1}{2}(A+B)$

So $\frac{1}{2}(A+B) = \frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}(+1). + 0.$

$\Rightarrow$ result.

Q 3 (d). By considering the value to which the series (∗) must converge at $x = 1/2$ deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

(2 marks)

At $x = \frac{1}{2}$, f is cont, with cont deriv., so

Series converges to $f(\frac{1}{2}) = \frac{1}{2}B$.

So.

$$\frac{1}{2}B = \frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cancel{\cos\left[(2m+1)\frac{\pi}{2}\right]}^{\,0}$$

$$- \left(\frac{A-B}{\pi}\right) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right).$$

$$= \frac{1}{4}(A+B) - \frac{A-B}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} \cdots\right)$$

$$\frac{1}{4}(B-A) = \frac{B-A}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} --\right)$$

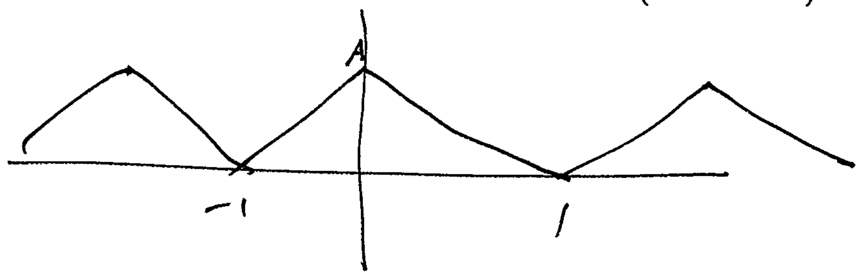
$\Rightarrow$ result

Question 3 continued on next page.

Q 3 (e). Describe and explain in a few words, with graphs, what happens to $f(x)$ and to the series $(*)$ when (a) $A = B$ and (b) $A = -B$.

(1.5 marks)

When $A = B$:

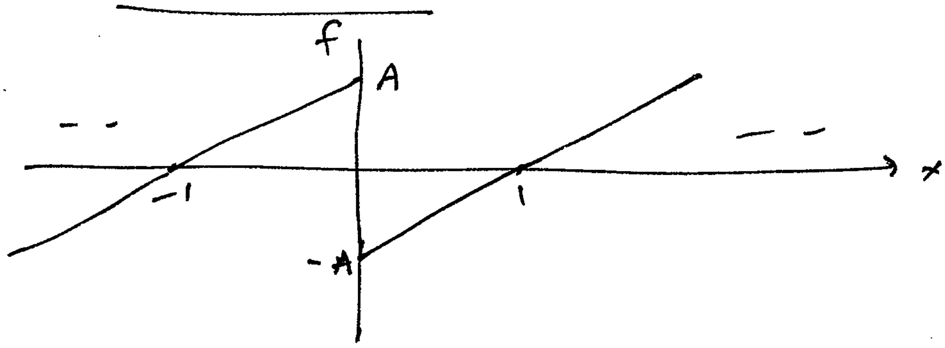


f is <u>even</u>.

F. Series is cosine series for f :

$$\frac{1}{2}A + \frac{4A}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos((2n+1)\pi x)$$

When $A = -B$



f is <u>odd</u>

F. Series is sine series for f :

$$-\frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x).$$

Question 4 see next page.

Q 4 (a). If

$$G(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-x^2/(4c^2 t)}$$

and if $G_t(x-y,t)$ denotes $\partial G(x-y,t)/\partial t$ *etc.*, check that

$$G_x(x-y,t) = -\frac{x-y}{2c^2 t}\, G(x-y,t)\,, \quad \text{and hence that}$$

$$G_{xx}(x-y,t)\left(= \partial\left(-\frac{x-y}{2c^2 t}\, G(x-y,t)\right)/\partial x\right) = \left(-\frac{1}{2c^2 t} + \frac{(x-y)^2}{4c^4 t^2}\right) G(x-y,t)\,.$$

Next check that

$$G_t(x-y,t) = \left(-\frac{1}{2t} + \frac{(x-y)^2}{4c^2 t^2}\right) G(x-y,t)$$

and so confirm that

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,.$$

(5.5 marks)

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-(x-y)^2/4c^2 t}$$

$$G_x(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}} \cdot \left[-\frac{2(x-y)}{4c^2 t}\right] e^{-(x-y)^2/4c^2 t}$$

$$= -\frac{x-y}{2c^2 t}\, G(x-y,t)$$

$$\Rightarrow G_{xx}(x-y,t) = -\frac{1}{2c^2 t} G(x-y,t) - \left(\frac{x-y}{2c^2 t}\right)\cdot\left(-\frac{x-y}{2c^2 t}\right) G(x-y,t)$$

$$= \left[-\frac{1}{2c^2 t} + \frac{(x-y)^2}{4c^4 t^2}\right] G(x-y,t)$$

$$G_t(x-y,t) = \frac{1}{\sqrt{4\pi c^2}} \cdot \frac{\partial}{\partial t}\left[t^{-\frac{1}{2}} e^{-(x-y)^2/4c^2 t}\right]$$

$$= \frac{1}{\sqrt{4\pi c^2}} \cdot \left\{ \left(-\tfrac{1}{2}\right) t^{-\frac{3}{2}} e^{-(x-y)^2/4c^2 t} + t^{-\frac{1}{2}} \frac{(x-y)^2}{4c^2 t^2}\, e^{-(x-y)^2/4c^2 t} \right\}$$

Question 4 continued on next page. TURN OVER

Q 4 (a). **Working space**

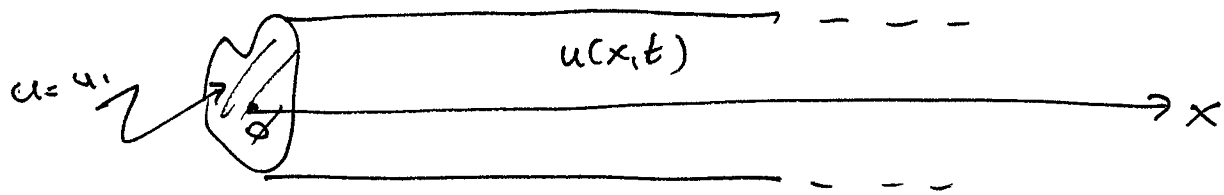
$$= -\frac{1}{2t} G(x-y,t) + \frac{(x-y)^2}{4c^2t^2} G(x-y,t)$$

$$\therefore \quad G_t(x-y,t) - c^2 G_{xx}(x-y,t) = 0$$

Question 4 continued on next page.

Q 4 (b). A very long (semi-infinite) cylindrical bar of material with thermal diffusivity c^2 lies along the positive x-axis. The sides of the bar are thermally insulated, and the temperature inside the bar is a function only of x and of time t. At time $t = 0$, the bar is at a uniform temperature u_0 along its length. The face of the bar at $x = 0$ is maintained for all $t > 0$ at the constant value u_1. Write down the partial differential equation, boundary condition and initial condition that determine the temperature distribution $u(x,t)$ in the bar for $t > 0$. Write down also the corresponding equations for the function $\hat{u}(x,t) = u(x,t) - u_1$.

(2.5 marks)



PDE: $u_t(x,t) = c^2 u_{xx}(x,t) \quad x > 0, \; t > 0$

B.C.: $u(0,t) = u_1, \quad t > 0$

I.C.: $u(x,0) = u_0, \quad x > 0.$

$\hat{u}(x,t) = u(x,t) - u_1$

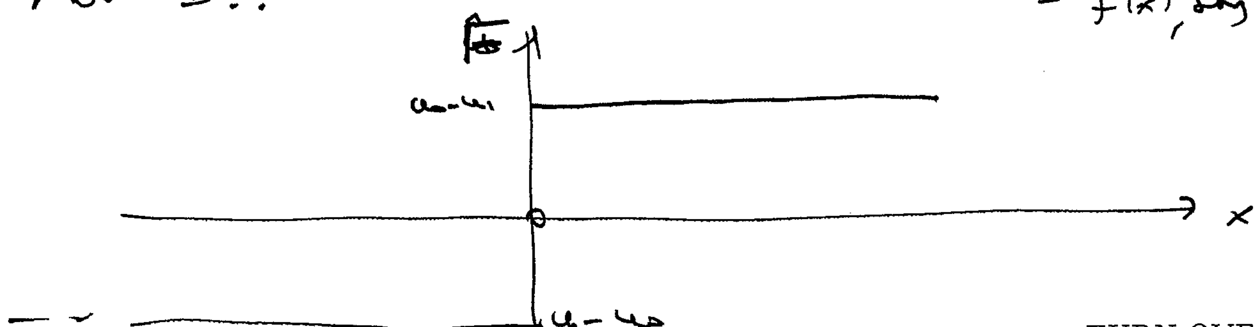
$\hat{u}_t(x,t) = u_t(x,t), \quad \hat{u}_{xx}(x,t) = u_{xx}(x,t)$

So new PDE: $\hat{u}_t(x,t) - c^2 \hat{u}_{xx}(x,t) = u_t - c^2 u_{xx} = 0, \quad x > 0, \; t > 0$

New B.C.: $\hat{u}(0,t) = u(0,t) - u_1 = 0, \quad t > 0$

New I.C.: $\hat{u}(x,0) = u(x,0) - u_1 = (u_0 - u_1), \quad x > 0$

$= f(x)$, say



Question 4 continued on next page.

Q 4 (c). **Given** the formula $u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$ for the temperature distribution that arises when $u(x,0) = f(x)$ for $-\infty < x < \infty$, in a bar that occupies the *whole* x-axis, deduce that the solution to the problem in Q 4(b) is

$$u(x,t) = u_1 + (u_0 - u_1)\ \mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right), \qquad (*)$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv\, .$$

(6 marks)

Extend $f(x)$ to an odd fn F, so B.C. for $\hat{u}$ will be satisfied automatically.

Then for $-\infty < x < \infty$, $t > 0$

$$\hat{u}(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, F(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_{-\infty}^{0} (u_1 - u_0)\, e^{-(x-y)^2/4c^2t}\, dy + \int_0^{\infty} (u_0 - u_1)\, e^{-(x-y)^2/4c^2t}\, dy \right\}$$

In each integral, put

$$\frac{x-y}{\sqrt{4c^2t}} = v \quad \Leftrightarrow \quad y = x - v\sqrt{4c^2t}, \quad dy = -\sqrt{4c^2t}\, dv$$

$y = -\infty \Leftrightarrow v = \infty$

$y = \infty \Leftrightarrow v = -\infty$

$y = 0 \Leftrightarrow v = \dfrac{x}{\sqrt{4c^2t}}$

Question 4 continued on next page.

Q 4 (c). **Working space**

So $$\hat{u}(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\left\{(u_1-u_0)\int_{\infty}^{x/\sqrt{4c^2t}} e^{-v^2}(-\sqrt{4c^2t})\,dv\right.$$

$$\left. + (u_0-u_1)\int_{x/\sqrt{4c^2t}}^{-\infty} e^{-v^2}(-\sqrt{4c^2t})\,dv\right\}$$

$$= \frac{1}{\sqrt{\pi}}\left\{(u_1-u_0)\int_{x/\sqrt{4c^2t}}^{\infty} e^{-v^2}\,dv + (u_0-u_1)\int_{-\infty}^{x/\sqrt{4c^2t}} e^{-v^2}\,dv\right\}$$

$$= \frac{1}{2}(u_1-u_0)\left\{\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-v^2}\,dv - \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4c^2t}} e^{-v^2}\,dv\right\}$$

$$+ \frac{1}{2}(u_0-u_1)\left\{\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-v^2}\,dv + \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4c^2t}} e^{-v^2}\,dv\right\}$$

$$= \frac{1}{2}(u_1-u_0)\left\{1-\operatorname{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)\right\}$$

$$+ \frac{1}{2}(u_0-u_1)\left\{1+\operatorname{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)\right\}$$

$$= (u_0-u_1)\operatorname{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)$$

This is also $\hat{u}$ soln for $x>0$ + B.C for $\hat{u}$

Then $u(x,t) = \hat{u}(x,t) + u_1$

$$= u_1 + (u_0-u_1)\operatorname{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)$$

Question 4 continued on next page.

Q 4 (d). Show how a measurement of $u_x(0,T)$ at some time $T > 0$ can be used to estimate T, if c^2 and $u_0 - u_1$ are known.

(2 marks)

$$u_x(x,t) = 0 + (u_0 - u_1)\left(\frac{1}{\sqrt{4c^2t}}\right)\frac{2}{\sqrt{\pi}} e^{-x^2/4c^2t}$$

$$\left[\text{using } \frac{d}{dz}\operatorname{erf}(z) = \frac{d}{dz}\frac{2}{\sqrt{\pi}}\int_0^z e^{-w^2}dw = \frac{2}{\sqrt{\pi}}e^{-z^2}\right]$$

So $$u_x(0,t) = \frac{(u_0 - u_1)}{\sqrt{\pi c^2 t}}$$

$$\sqrt{t} = \frac{(u_0 - u_1)}{\sqrt{\pi c^2}\; u_x(0,t)}$$

So at time T.

$$T = \frac{(u_0 - u_1)^2}{\pi c^2 [u_x(0,T)]^2}$$

Table of Laplace Transforms

$f(t)$	$F(s)$
K	$\dfrac{K}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\dfrac{\alpha}{s^2+\alpha^2}$
$e^{at}f(t)$	$F(s-a)$
$f(t-k)u(t-k) = \left\{ \begin{array}{cc} 0 & t<k \\ f(t-k) & k \leq t \end{array} \right]$	$e^{-ks}F(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$