



THE UNIVERSITY OF QUEENSLAND

St Lucia Campus

STUDENT NAME:

Fred Bloggs

STUDENT NUMBER:

S	1	2	3	4	5	6	7	8
---	---	---	---	---	---	---	---	---

SEAT NUMBER:

FINAL Examination Second Semester, 2006

COURSE CODE MATH2100

COURSE TITLE Applied Mathematical Analysis

EXAMINER Prof. A.J. Bracken

WEIGHTING/MARKS 65

PERUSAL TIME 10 mins No writing permitted during perusal

WRITING TIME 2 Hrs 0 mins

NO. OF PAGES IN EXAM PAPER (including cover sheet) 21

Exam Type:

Closed Book

Calculator:

Yes - ~~Non-programmable~~

Dictionary:

No

Permitted Materials:

but all memory to be cleared.

SPECIAL INSTRUCTIONS TO STUDENTS:

Answer all four questions. Marks for each part question are shown after that part. The total marks available are 65. Your score on the paper will be added to your score out of 35 for the assignments to give your final mark out of 100.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

General Award Rules 1A.7 and 1A.8

1A.7 Responsibilities of students

- 1A.7.1 A student must comply with the examination instructions and directions given by an examination supervisor.
- 1A.7.2 A student may not enter an examination room without the permission of the examination supervisor, or after the first 10 minutes of examination working time.
- 1A.7.3 A student must not leave the examination room without the permission of the examination supervisor.
- 1A.7.4 Permission will not be granted under GAR 1A.7.3 during —
 - (a) the first 10 minutes of examination working time; and
 - (b) the final 10 minutes.
- 1A.7.5
 - (a) A student must bring into the examination some identification in the form of a current student card or other photographic identification.
 - (b) The identification must be displayed throughout the examination.
 - (c) Before the start of the examination, a student without identification must sign a declaration in a form set by the secretary and registrar.
 - (d) A student who does an examination without identification must produce identification at a location specified in writing by the secretary and registrar either generally or for that student.
 - (e) The university may withhold the results for an examination for a student who did not have identification at the examination until the student has produced identification under GAR 1A.7.5(d).
- 1A.7.6 Unless addressing a question to the examiner or examination supervisor, a student must not communicate in any way with another person during the examination.
- 1A.7.7 A student may bring unauthorised material into the examination room only if the material —
 - (a) is brought in with the permission of the examiner or examination supervisor; or
 - (b) is left with the examination supervisor immediately on entering the examination room.
- 1A.7.8 A student may remove examination books, scripts or material provided to the student during the examination only with the permission of the examination supervisor.

1A.8 Examination supervisors

- 1A.8.1 The examination supervisor may —
 - (a) inspect any material brought into the examination room by a student; and
 - (b) confiscate any material which the examination supervisor reasonably suspects to be or to contain unauthorised material.
- 1A.8.2 If the examination supervisor reasonably believes that a student's behaviour may distract or disturb other students, the examination supervisor may direct the student to leave the examination room.

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

- Q1. (a) *Find the general solution of:* Sketch the trajectories of the following system in the phase plane, indicating the direction of flow, and classify the type and stability of the critical point at the origin.

$p = -6, q = 9 - 4 = 5, \Delta = p^2 - 4q = 16$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \underline{\dot{\mathbf{y}}} = \underline{A} \underline{\mathbf{y}}$$

State the equation of any straight line trajectories.
Determine the slope of the trajectories on the y_1 and y_2 axes and indicate your results on your sketch. (8.5 marks)

improper stable, attractive node

$A = \begin{bmatrix} -3 & 1 \\ 4 & -3 \end{bmatrix}$

E' Values: $\begin{vmatrix} -3-\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (\lambda + 3)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + 5 = 0 \Rightarrow (\lambda + 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -1, -5$$

E' Vectors: $\lambda = -1:$

$$\begin{bmatrix} -3+1 & 1 \\ 4 & -3+1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2u + v = 0 \\ 4u - 2v = 0 \end{cases} \Rightarrow v = 2u$$

Take $u = 1 \Rightarrow v = -2$, so $\underline{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = -5:$ $\begin{bmatrix} -3+5 & 1 \\ 4 & -3+5 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2u + v = 0 \\ 4u + 2v = 0 \end{cases} \Rightarrow v = -2u$

Take $u = 1$ so $v = -2$ so $\underline{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Gen. sol:

$$\underline{\mathbf{y}} = A \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}$$

If $B = 0:$

$$\underline{\mathbf{y}}_1 = A e^{-t}, \underline{\mathbf{y}}_2 = 2A e^{-t} \Rightarrow \underline{\mathbf{y}}_2 = 2 \underline{\mathbf{y}}_1$$

If $A = 0:$

$$\underline{\mathbf{y}}_1 = B e^{-5t}, \underline{\mathbf{y}}_2 = -2B e^{-5t} \Rightarrow \underline{\mathbf{y}}_2 = -2 \underline{\mathbf{y}}_1$$

- See as $t \rightarrow \infty, \underline{\mathbf{y}} \rightarrow \underline{\mathbf{0}}$ in each case

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 1 (a). Working space only

Slope equation: $\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{4y_1 - 3y_2}{-3y_1 + y_2}$

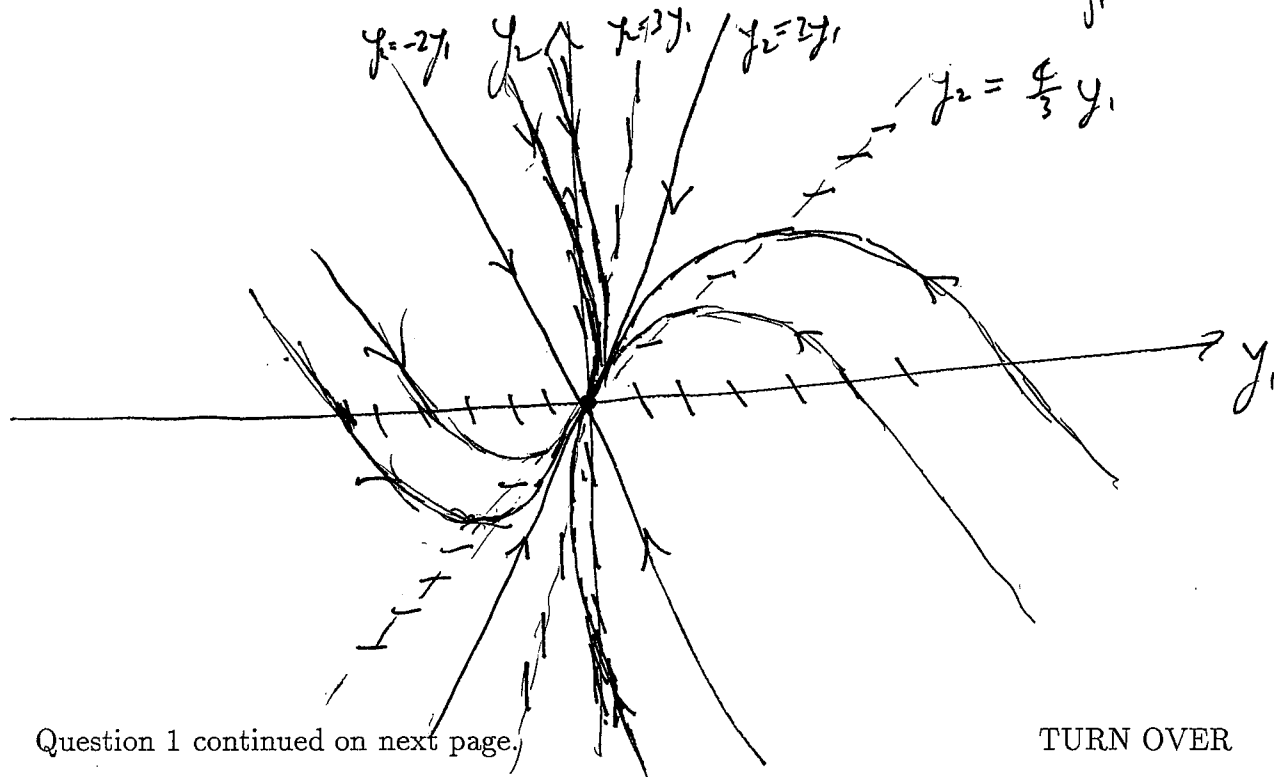
$$\frac{dy_2}{dy_1} = 0 \Leftrightarrow 4y_1 = 3y_2 \quad \text{or} \quad y_2 = \frac{4}{3}y_1$$

$$\frac{dy_2}{dy_1} = \pm \infty \Leftrightarrow -3y_1 + y_2 = 0 \quad \text{or} \quad y_2 = 3y_1$$

When $y_2 = 0$ (y_1 axis) $\frac{dy_2}{dy_1} = -\frac{4}{3}$

When $y_1 = 0$ (y_2 axis) $\frac{dy_2}{dy_1} = -3$

Directions: When $y_2 = 0$, $\frac{dy_1}{dt} = -3y_1$, so y_1 inc $\Leftrightarrow y_1 < 0$
 y_1 dec $\Leftrightarrow y_1 > 0$



MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q1. (b) Find all the **critical points** for the following nonlinear system.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_1y_2 - 4y_1^2 \\ 3y_2 + y_1y_2 - y_2^2 \end{pmatrix}$$

$$\begin{aligned} y_1 &\equiv r \\ y_2 &\equiv s \end{aligned}$$

Then use Linearization to find the type and stability of the critical points that do not lie on the r axis, i.e. for which $s \neq 0$.

(9 marks)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{pmatrix}$$

$$F_1(y_1, y_2) = 2y_1 + 3y_1y_2 - 4y_1^2$$

$$F_2(y_1, y_2) = 3y_2 + y_1y_2 - y_2^2$$

Critical points: $F_1 = 0$ and $F_2 = 0$.

$$y_1(2 + 3y_2 - 4y_1) = 0 \quad \text{and} \quad y_2(3 + y_1 - y_2) = 0$$

For Critical points with $y_2 \neq 0$,

$$3 + y_1 - y_2 = 0$$

$$\text{e. } y_2 = y_1 + 3$$

and either $y_1 = 0 \Rightarrow y_2 = 3$

$$\text{or } 2 + 3y_2 - 4y_1 = 0$$

$$\Rightarrow 2 + 3(y_1 + 3) - 4y_1 = 0$$

$$\Rightarrow y_1 = 11 \Rightarrow y_2 = 14$$

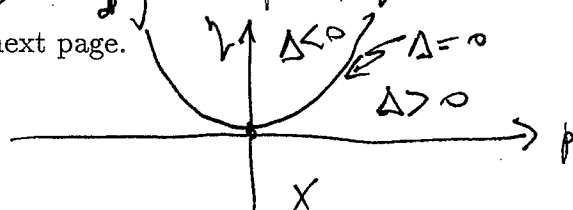
So critical points are $(0, 3)$ and $(11, 14)$.

$$F = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 2 + 3y_2 - 8y_1 & 3y_1 \\ y_2 & 3 + y_1 - 2y_2 \end{bmatrix}$$

At $(0, 3)$, $\dot{\underline{y}} \approx F \underline{y}$ where $\underline{y} = \underline{y} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$F = \begin{bmatrix} 11 & 0 \\ 3 & -3 \end{bmatrix} \Rightarrow p = 8, q = -33, \Delta = 64 - 132 = -68$$

Question 1 continued on next page.



TURN OVER

Saddle pt.
unstable

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 1 (b). Working space only

At $(11, 14)$, $\dot{\underline{y}} \approx F \underline{y}$ where $\underline{y} = \begin{pmatrix} y \\ 14 \end{pmatrix}$

$F = \begin{bmatrix} 44-88 & 33 \\ 14 & 14-28 \end{bmatrix} = \begin{bmatrix} -44 & 33 \\ 14 & -14 \end{bmatrix}$

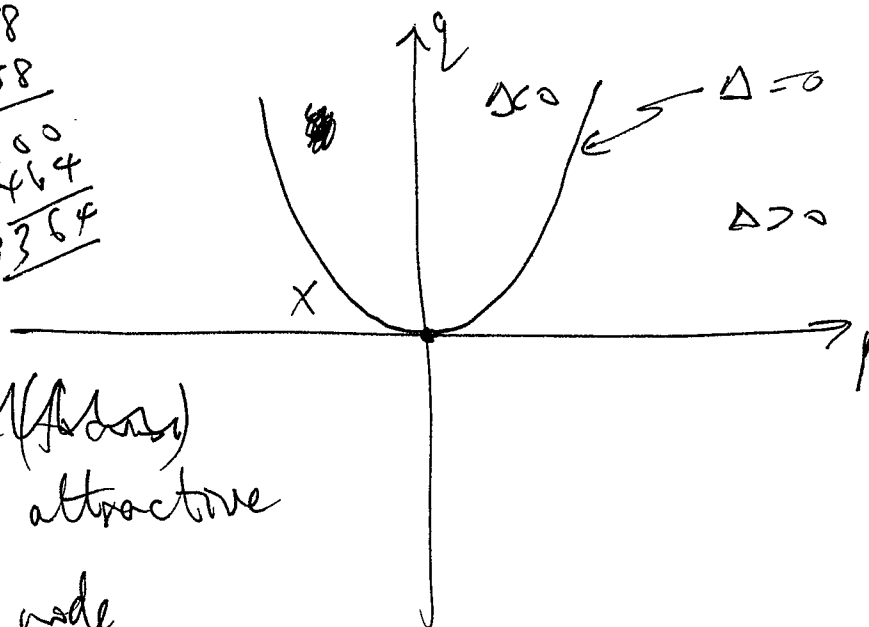
$\Rightarrow p = -58, q = (44)(-14) - (14)(33) = (-44)(11)(14) = 154$

$\Delta = p^2 - 4q = (58)^2 - 4(154) = 3364 - 616 = 2748$

$$\begin{array}{r} 86 \\ 186 \\ 368 \\ 516 \\ 4196 \end{array}$$

$$\begin{array}{r} 99 \\ 498 \\ 696 \\ 594 \\ 5544 \end{array}$$

$$\begin{array}{r} 58 \\ 58 \\ 2900 \\ 464 \\ 3364 \end{array}$$



Attractively stable (focus)
Stable & attractive
Improper node

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q2. (a) If $G(s)$ is the Laplace Transform of $g(t)$, prove carefully that

$$\frac{dG(s)}{ds} \quad \text{is the Laplace Transform of} \quad -tg(t).$$

Hence, or otherwise, determine the inverse Laplace Transform of

$$F(s) = \frac{2s}{(s^2 + 1)^2}.$$

$$\mathcal{L}\{tg(t)\} = \mathcal{L}\left\{t \int_0^\infty g(t) e^{-st} dt\right\}$$

(6 marks)

$$\begin{aligned} G(s) &= \int_0^\infty e^{-st} g(t) dt \\ \Rightarrow \frac{dG(s)}{ds} &= \int_0^\infty \frac{d}{ds} (e^{-st}) g(t) dt = \int_0^\infty -t e^{-st} g(t) dt \\ &= \mathcal{L}(-tg(t)). \end{aligned}$$

$$F(s) = \frac{2s}{(s^2+1)^2} = \frac{d}{ds} \left[\frac{-1}{(s^2+1)} \right]$$

$$\text{Now } \frac{-1}{s^2+1} = \mathcal{L}(-\sin t)$$

$$\begin{aligned} \text{So } \frac{2s}{(s^2+1)^2} &= \mathcal{L}(-t \sin(-t)) \\ &= \mathcal{L}(t \sin t) \end{aligned}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q2. (b) Use Laplace Transforms to solve the following initial value problem.

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = f(t) = \begin{cases} 0 & t < 3 \\ e^{-2t} & 3 \leq t \end{cases} \quad \text{with } y(0) = 0 \text{ and } \frac{dy}{dt}(0) = 0$$

Let $Y(s) = \mathcal{L}(y(t))$

(9 marks)

Then $\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$

$$\mathcal{L}(y''(t)) = s^2 Y(s) - sy'(0) - y'(0) = s^2 Y(s)$$

$$\begin{aligned} \text{Also } \mathcal{L}(f(t)) &= \mathcal{L}(e^{-2t} u(t-3)) \\ &= e^{-6} \mathcal{L}(e^{-2(t-3)} u(t-3)) \\ &= e^{-6} \frac{e^{-3s}}{s+2} \end{aligned}$$

So

$$s^2 Y + 5sY + 6Y = e^{-6} \frac{e^{-3s}}{s+2}$$

$$(s+3)(s+2) Y =$$

$$Y = e^{-6} e^{-3s} \frac{1}{(s+3)(s+2)^2}$$

Now

$$\frac{1}{(s+3)(s+2)^2} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+3)(s+2) + C(s+3)$$

$$s = -2 \Rightarrow 1 = 0 + 0 + C \Rightarrow C = 1$$

$$s = -3 \Rightarrow 1 = A + 0 + 0 \Rightarrow A = 1$$

$$\text{Coeff of } s^2: 0 = A + B \Rightarrow B = -A = -1$$

$$\text{So } Y = e^{-6} e^{-3s} \left[\frac{1}{s+3} - \frac{1}{s+2} + \frac{1}{(s+2)^2} \right]$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 2 (b). Working space only

$$\begin{aligned}
 \text{Then } y(t) &= e^{-t} \left[e^{-3(t-3)} - e^{-2(t-3)} + (t-3)e^{-2(t-3)} \right] u(t-3) \\
 &= \left[e^3 e^{-3t} - e^{-2t} + (t-3)e^{-2t} \right] u(t-3) \\
 &= \begin{cases} e^3 e^{-3t} + e^{-2t}(t-4) & t \geq 3 \\ 0 & t < 3 \end{cases}
 \end{aligned}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 3 (a). Use integration by parts to show that if p is a nonzero constant, then

$$\int x \cos(px) dx = \frac{x}{p} \sin(px) + \frac{1}{p^2} \cos(px) + \text{const.},$$

$$\int x \sin(px) dx = -\frac{x}{p} \cos(px) + \frac{1}{p^2} \sin(px) + \text{const.}.$$

(2.5 marks)

$$\begin{aligned} \int x \cos(px) dx &= \frac{x \sin(px)}{p} - \frac{1}{p} \int \sin(px) dx \\ &= \frac{x \sin(px)}{p} + \frac{1}{p^2} \cos(px) + C \end{aligned}$$

$$\begin{aligned} \int x \sin(px) dx &= -\frac{x \cos(px)}{p} + \frac{1}{p} \int \cos(px) dx \\ &= -\frac{x \cos(px)}{p} + \frac{1}{p^2} \sin(px) + C \end{aligned}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 3 (b). Show that the Fourier Series corresponding to the function defined by

$$f(x) = A(1+x), \quad -1 \leq x \leq 0; \quad f(x) = B(1-x), \quad 0 < x < 1;$$

and $f(x+2) = f(x), \quad -\infty < x < \infty,$

$p=2, L=1$

where A and B are constants, is given by

$$\frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos([2m+1]\pi x) - \frac{(A-B)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x). \quad (*)$$

Series is $a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$

(8.5 marks)

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2} A \int_{-1}^0 (1+x) dx + \frac{1}{2} B \int_0^1 (1-x) dx \\ &= \frac{1}{2} A \left[x + \frac{1}{2} x^2 \right]_{-1}^0 + \frac{1}{2} B \left[x - \frac{1}{2} x^2 \right]_0^1 \\ &= \frac{1}{2} A [0 - (-1 + \frac{1}{2})] + \frac{1}{2} B [(1 - \frac{1}{2}) - 0] \\ &= \frac{1}{4} (A+B) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = A \int_{-1}^0 (1+x) \cos(n\pi x) dx + B \int_0^1 (1-x) \cos(n\pi x) dx \\ &= A \left\{ \left[\frac{1+x}{n\pi} \sin(n\pi x) \right]_{-1}^0 - \frac{1}{n\pi} \int_{-1}^0 \sin(n\pi x) dx \right\} \\ &\quad + B \left\{ \left[\frac{1-x}{n\pi} \sin(n\pi x) \right]_0^1 + \frac{1}{n\pi} \int_0^1 \sin(n\pi x) dx \right\} \\ &= + \frac{A}{(n\pi)^2} [\cos(n\pi x)]_{-1}^0 - \frac{B}{(n\pi)^2} [\cos(n\pi x)]_0^1 \\ &= + \frac{A}{(n\pi)^2} [1 - (-1)^n] - \frac{B}{(n\pi)^2} [(-1)^n - 1] \\ &= \frac{A+B}{(n\pi)^2} [1 - (-1)^n] = \begin{cases} \frac{2(A+B)}{(2m+1)\pi^2} & n = 2m+1 \text{ odd} \\ 0 & n = 2m \text{ even} \end{cases} \end{aligned}$$

$m=0, 1, 2, \dots$
 $m=1, 2, \dots$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 3 (b). Working space

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x) dx = A \int_{-1}^0 (1+x) \sin(n\pi x) dx + B \int_0^1 (1-x) \sin(n\pi x) dx$$

$$= A \left\{ \left[-\frac{(1+x)}{n\pi} \cos(n\pi x) \right]_{-1}^0 + \frac{1}{n\pi} \int_{-1}^0 \cos(n\pi x) dx \right\} \\ + B \left\{ \left[-\frac{(1-x)}{n\pi} \cos(n\pi x) \right]_0^1 - \frac{1}{n\pi} \int_0^1 \cos(n\pi x) dx \right\}$$

$$= \frac{A}{n\pi} [-1 - 0] + \frac{A}{n^2\pi^2} [\sin(n\pi x)]_{-1}^0 \\ + \frac{B}{n\pi} [0 + 1] - \frac{B}{n^2\pi^2} [\sin(n\pi x)]_0^1$$

$$= - \frac{(A - B)}{n\pi}.$$

∴ series is

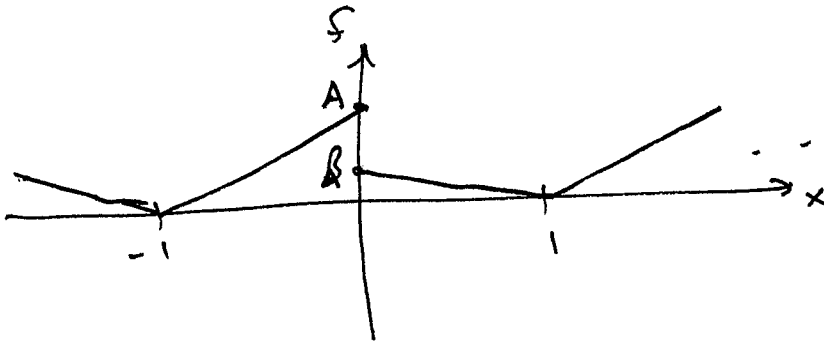
$$\frac{1}{4} (A+B) + \frac{2(A+B)}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos[(2m+1)\pi x]$$

$$- \frac{(A-B)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 3 (c). To what value must the series (*) converge at $x = 0$? Use this result to deduce that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$



(2 marks)

f is piecewise continuous and $f(0-), f(0+), f'(0-), f'(0+)$ well-defined.

Series converges to $\frac{1}{2}(A+B)$

$$\frac{1}{2}(A+B) = \frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} (+1) + 0.$$

→ result.

Q 3 (d). By considering the value to which the series (*) must converge at $x = 1/2$ deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}.$$

(2 marks)

At $x = \frac{1}{2}$, f is continuous, with continuous derivative, so

Series converges to $f(\frac{1}{2}) = \frac{1}{2}B$.

$$\begin{aligned} \text{So } \frac{1}{2}B &= \frac{1}{4}(A+B) + \frac{2(A+B)}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2}\right) \\ &\quad - \left(\frac{A-B}{\pi}\right) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

$$= \frac{1}{4}(A+B) - \frac{A-B}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

$$\frac{1}{4}(B-A) = \frac{B-A}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

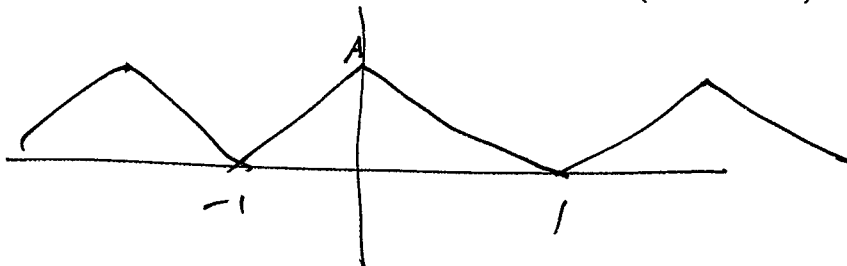
→ result

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 3 (e). Describe and explain in a few words, with graphs, what happens to $f(x)$ and to the series (*) when (a) $A = B$ and (b) $A = -B$.

(1.5 marks)

When $A = B$:

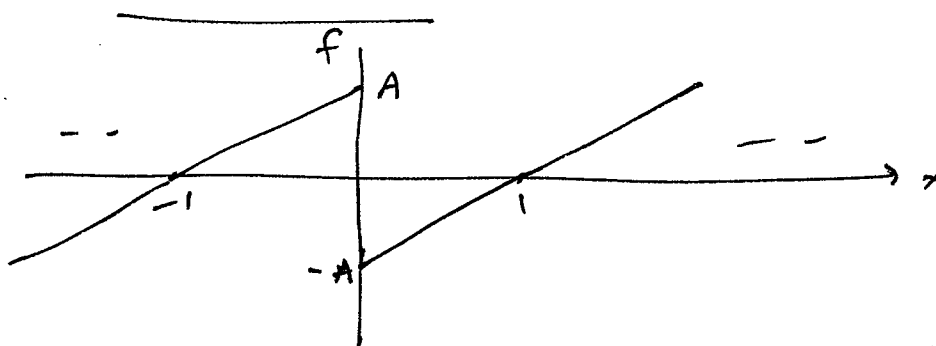


f is even.

F. Series is cosine series for f :

$$\frac{1}{2} A + \frac{4A}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \cos((n+1)\pi x)$$

When $A = -B$



f is odd

F. Series is sine series for f :

$$- \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x)$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (a). If

$$G(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-x^2/(4c^2 t)}$$

and if $G_t(x - y, t)$ denotes $\partial G(x - y, t)/\partial t$ etc., check that

$$G_x(x - y, t) = -\frac{x - y}{2c^2 t} G(x - y, t), \text{ and hence that}$$

$$G_{xx}(x - y, t) \left(= \partial \left(-\frac{x - y}{2c^2 t} G(x - y, t) \right) / \partial x \right) = \left(-\frac{1}{2c^2 t} + \frac{(x - y)^2}{4c^4 t^2} \right) G(x - y, t).$$

Next check that

$$G_t(x - y, t) = \left(-\frac{1}{2t} + \frac{(x - y)^2}{4c^2 t^2} \right) G(x - y, t)$$

and so confirm that

$$G_t(x - y, t) = c^2 G_{xx}(x - y, t).$$

(5.5 marks)

$$G(x, y, t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-(x-y)^2/4c^2 t}$$

$$G_x(x-y, t) = \frac{1}{\sqrt{4\pi c^2 t}} \cdot \left[-\frac{2(x-y)}{4c^2 t} \right] e^{-(x-y)^2/4c^2 t}$$

$$= -\frac{x-y}{2c^2 t} G(x-y, t)$$

$$\Rightarrow G_{xx}(x-y, t) = -\frac{1}{2c^2 t} G(x-y, t) - \left(\frac{x-y}{2c^2 t} \right) \cdot \left(-\frac{x-y}{2c^2 t} \right) G(x-y, t)$$

$$= \left[-\frac{1}{2c^2 t} + \frac{(x-y)^2}{4c^4 t^2} \right] G(x-y, t)$$

$$G_t(x-y, t) = \frac{1}{\sqrt{4\pi c^2}} \cdot \frac{\partial}{\partial t} \left[t^{-\frac{1}{2}} e^{-(x-y)^2/4c^2 t} \right]$$

$$= \frac{1}{\sqrt{4\pi c^2}} \cdot \left\{ (-\frac{1}{2}) t^{-\frac{3}{2}} e^{-(x-y)^2/4c^2 t} + t^{-\frac{1}{2}} \left(\frac{(x-y)^2}{4c^2 t} \right) e^{-(x-y)^2/4c^2 t} \right\}$$

Question 4 continued on next page.

TURN OVER

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (a). Working space

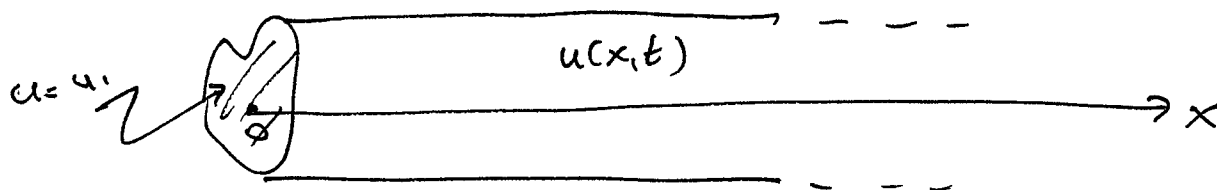
$$= -\frac{1}{2t} G(x-y, t) + \frac{(x-y)^2}{4c^2 t^2} G(x-y, t)$$

$$\therefore \underline{G_t(x-y, t) - c^2 G_{xx}(x-y, t) = 0}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (b). A very long (semi-infinite) cylindrical bar of material with thermal diffusivity c^2 lies along the positive x -axis. The sides of the bar are thermally insulated, and the temperature inside the bar is a function only of x and of time t . At time $t = 0$, the bar is at a uniform temperature u_0 along its length. The face of the bar at $x = 0$ is maintained for all $t > 0$ at the constant value u_1 . Write down the partial differential equation, boundary condition and initial condition that determine the temperature distribution $u(x, t)$ in the bar for $t > 0$. Write down also the corresponding equations for the function $\hat{u}(x, t) = u(x, t) - u_1$.

(2.5 marks)



PDE: $u_t(x, t) = c^2 u_{xx}(x, t) \quad x > 0, t > 0$

B.C.: $u(0, t) = u_1, \quad t > 0$

I.C.: $u(x, 0) = u_0, \quad x > 0.$

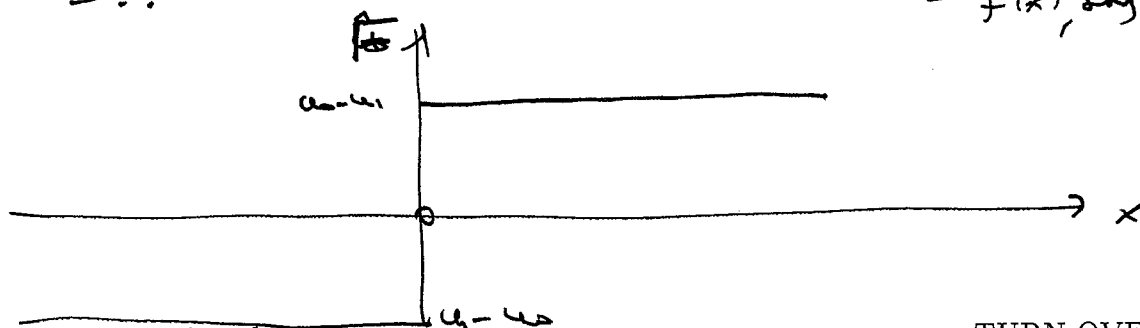
$\hat{u}(x, t) = u(x, t) - u_1$

$\hat{u}_t(x, t) = u_t(x, t), \quad \hat{u}_{xx}(x, t) = u_{xx}(x, t)$

New PDE: $\hat{u}_t(x, t) - c^2 \hat{u}_{xx}(x, t) = 0, \quad x > 0, t > 0.$

New B.C.: $\hat{u}(0, t) = u(0, t) - u_1 = 0, \quad t > 0$

New I.C.: $\hat{u}(x, 0) = u(x, 0) - u_1 = (u_0 - u_1), \quad x > 0$
 $= f(x), \text{ say}$



MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (c). Given the formula $u(x, t) = \int_{-\infty}^{\infty} G(x-y, t) f(y) dy$ for the temperature distribution that arises when $u(x, 0) = f(x)$ for $-\infty < x < \infty$, in a bar that occupies the whole x -axis, deduce that the solution to the problem in Q 4(b) is

$$u(x, t) = u_1 + (u_0 - u_1) \operatorname{erf} \left(\frac{x}{\sqrt{4c^2 t}} \right), \quad (*)$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2} dv.$$

(6 marks)

Extend $f(x)$ to an odd fun^F P.C. for
it will be satisfied automatically.

Then for $-\infty < x < \infty, t > 0$

$$\hat{u}(x, t) = \int_{-\infty}^{\infty} G(x-y, t) F(y) dy$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_{-\infty}^0 (u_1 - u_0) e^{-(x-y)^2/4c^2 t} dy \right.$$

$$\left. + \int_0^{\infty} (u_0 - u_1) e^{-(x-y)^2/4c^2 t} dy \right\}$$

In each integral,

$$\frac{x-y}{\sqrt{4c^2 t}} = v$$

$$\Leftrightarrow y = x - v\sqrt{4c^2 t}$$

$$dy = -\sqrt{4c^2 t} dv$$

$$y = -\infty \Leftrightarrow v = \infty$$

$$y = \infty \Leftrightarrow v = -\infty$$

$$y = 0 \Leftrightarrow v = \frac{x}{\sqrt{4c^2 t}}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (c). Working space

$$\begin{aligned}
 \hat{u}(x,t) &= \frac{1}{\sqrt{4\kappa ct}} \left\{ (u_1 - u_0) \int_{-\infty}^{x/\sqrt{4\kappa ct}} e^{-v^2} (\sqrt{4\kappa ct}) dv \right. \\
 &\quad \left. + (u_0 - u_1) \int_{x/\sqrt{4\kappa ct}}^{\infty} e^{-v^2} (-\sqrt{4\kappa ct}) dv \right\} \\
 &= \frac{1}{\sqrt{\kappa}} \left\{ (u_1 - u_0) \int_{x/\sqrt{4\kappa ct}}^{\infty} e^{-v^2} dv + (u_0 - u_1) \int_{-\infty}^{x/\sqrt{4\kappa ct}} e^{-v^2} dv \right\} \\
 &= \frac{1}{2}(u_1 - u_0) \left\{ \frac{2}{\sqrt{\kappa}} \int_0^{\infty} e^{-v^2} dv - \frac{2}{\sqrt{\kappa}} \int_0^{x/\sqrt{4\kappa ct}} e^{-v^2} dv \right\} \\
 &\quad + \frac{1}{2}(u_0 - u_1) \left\{ \frac{2}{\sqrt{\kappa}} \int_{-\infty}^0 e^{-v^2} dv + \frac{2}{\sqrt{\kappa}} \int_0^{x/\sqrt{4\kappa ct}} e^{-v^2} dv \right\} \\
 &= \frac{1}{2}(u_1 - u_0) \left\{ 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa ct}}\right) \right\} \\
 &\quad + \frac{1}{2}(u_0 - u_1) \left\{ 1 + \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa ct}}\right) \right\} \\
 &= (u_0 - u_1) \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa ct}}\right)
 \end{aligned}$$

This is also \hat{u}_{erf} for $x > 0$ + B.C. for \hat{u}
 Then $u(x,t) = \hat{u}(x,t) + u_1$

$$= u_1 + (u_0 - u_1) \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa ct}}\right)$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Q 4 (d). Show how a measurement of $u_x(0, T)$ at some time $T > 0$ can be used to estimate T , if c^2 and $u_0 - u_1$ are known.

(2 marks)

$$u_x(x, t) = 0 + (u_0 - u_1) \left(\frac{1}{\sqrt{4c^2 t}} \right) \frac{2}{\sqrt{\pi}} e^{-x^2/4c^2 t}$$

$$\left[\text{using } \frac{d}{dz} \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw = \frac{2}{\sqrt{\pi}} e^{-z^2} \right]$$

$$\int u_x(0, t) = \frac{(u_0 - u_1)}{\sqrt{\pi c^2 t}}$$

$$\sqrt{t} = \frac{(u_0 - u_1)}{\sqrt{\pi c^2} u_x(0, t)}$$

So at time T .

$$T = \frac{(u_0 - u_1)^2}{\pi c^2 [u_x(0, T)]^2}$$

MATH2100 — APPLIED MATHEMATICAL ANALYSIS
Second Semester Examination, November, 2006 (continued)

Table of Laplace Transforms

$f(t)$	$F(s)$
K	$\frac{K}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\alpha t)$	$\frac{s}{s^2 + \alpha^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2 + \alpha^2}$
$e^{at}f(t)$	$F(s-a)$
$f(t-k)u(t-k) = \begin{cases} 0 & t < k \\ f(t-k) & k \leq t \end{cases}$	$e^{-ks}F(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$