INTERNAL STUDENTS ONLY THE UNIVERSITY OF QUEENSLAND

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Second Semester Examination, November, 2004

MATH2010

ANALYSIS OF ORDINARY DIFFERENTIAL EQUATIONS

(Unit Courses, Inf. Tech.)

Time: ONE HOUR for working

Ten minutes for perusal before examination begins

Check that this examination paper has 9 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!
Students should attempt **all** questions.
The exam paper is a total of 65 marks
The marks allocated to each part of each question are as indicated.
Calculators allowed, but all memory must be cleared beforehand.
Check that this examination paper has 9 printed pages!

FAMILY NAME (PRINT): ________________

GIVEN NAMES (PRINT): ________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		2	
TOTAL MARKS			

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Q1. (a) **Sketch the trajectories of the following system in the phase plane**, indicating the direction of flow, and **classify the type and stability** of the fixed point at the origin.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

State the equation of any straight line trajectories.
State (briefly) why the direction of flow is as you have indicated.

(17 marks)

Solution Either calculate the determinant of the matrix, which is -2, or find the eigenvalues and vectors of the matrix, which are $\lambda = 2, -1$, both of which implies the critical point at the origin is a **saddle**, which is **unstable**. To sketch a saddle you need to find the eigenvalues and eigenvectors of the matrix. The eigenvectors give the straightline solutions. Here

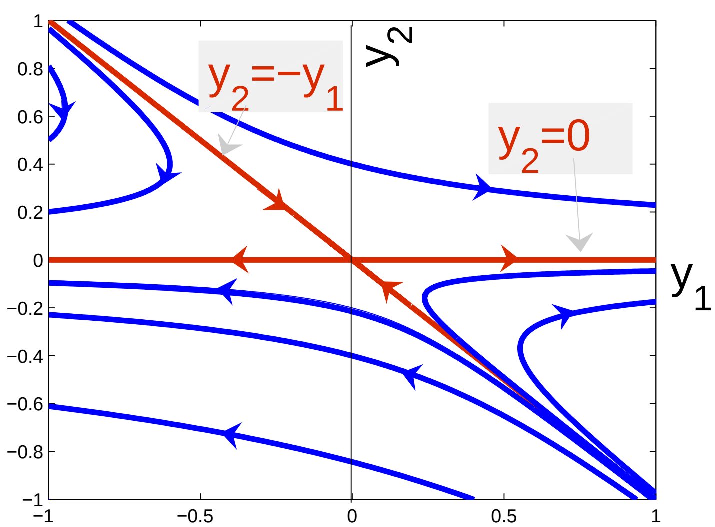
$$\lambda = 2, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda = -1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So that the straightline solutions are

$$y_2 = 0 \text{ from } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on which } y_1 = c_1 e^{2t} \text{ exponential growth}$$

$$y_2 = -y_1 \text{ from } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ on which } y_2 = -y_1 = -c_2 e^{-t} \text{ exponential decay}$$

For exponential growth the direction of flow is away from the critical point. For exponential decay the direction of flow is towards the critical point.



Question 1 continued on next page.

Q1. (b) Find all the **critical points** for the following nonlinear system.

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 3r - rs - r^2 \\ -s + 2rs \end{pmatrix}$$

Then use Linearization to find the type and stability of the critical point **that does not** lie on an axis, i.e. for which $r \neq 0$ AND $s \neq 0$.

(16 marks)

Solution The critical points are given by $\dot{r} = 0$ AND $\dot{s} = 0$.
Here that is

$$3r - rs - r^2 \quad \text{AND} \quad -s + 2rs$$

$$\{r = 0 \quad \text{or} \quad 3 - s - r = 0\} \qquad \text{AND} \qquad \{s = 0 \quad \text{or} \quad -1 + 2r = 0\}$$

So the 4 cases are

$$r = 0 \quad \text{AND} \quad s = 0 \quad \Rightarrow (0,\, 0)$$

$$r = 0 \quad \text{AND} \quad -1 + 2r = 0 \quad \text{NO solution}$$

$$3 - s - r = 0 \quad \text{AND} \quad s = 0 \quad \Rightarrow (3,\, 0)$$

$$3 - s - r = 0 \quad \text{AND} \quad -1 + 2r = 0 \quad \Rightarrow (\tfrac{1}{2},\, 2\tfrac{1}{2})$$

So the critical points are $(0,\, 0)$, $(3,\, 0)$ and $(\frac{1}{2},\, 2\frac{1}{2})$.

The only critical point **NOT** on an axis is $(\frac{1}{2},\, 2\frac{1}{2})$.
Now

$$\mathbf{Df} = \begin{pmatrix} 3 - s - 2r & -r \\ 2s & -1 + 2r \end{pmatrix}$$

So that

$$\mathbf{Df}(\tfrac{1}{2},\, 2\tfrac{1}{2}) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\det\mathbf{Df} = \frac{1}{2} > 0, \ \text{trace}\mathbf{Df} = -\frac{1}{2}, \ \text{trace}^2 - 4\det = \frac{1}{4} - \frac{4}{2} < 0$$

So the critical point at $(\frac{1}{2},\, 2\frac{1}{2})$ is a **stable spiral**.

Q2. (a) Prove that the Laplace Transform of

$$f(t) = \left\{ \begin{array}{cc} t & t < 3 \\ 3 & 3 \leq t \end{array} \right] \quad \text{is} \quad F(s) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$

State clearly any theorems that you use. **(8 marks)**

Solution

First rewrite the function $f(t)$ in terms of step functions:

$$f(t) = t(1 - u(t-3)) + 3u(t-3) = t - (t-3)u(t-3)$$

Now $L(t) = \dfrac{1}{s^2}$ and using the Second Shifting Theorem

$$L\left(g(t-k)u(t-k)\right) = e^{-ks}G(s) \quad \text{with} \quad g(t-3) = t-3 \Rightarrow g(t) = t$$

gives $L((t-3)u(t-3)) = \dfrac{e^{-3s}}{s^2}$. Finally

$$L(f(t)) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}.$$

Alternatively you can use the definition of the Laplace Transform and integrate directly:

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt = \int_0^3 te^{-st}dt + \int_3^\infty 3e^{-st}dt$$

Now to integrate te^{-st} use integration by parts with $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = e^{-st}/(-s)$.

$$\int te^{-st}dt = \frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s}dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

Adding the limits:

$$\int_0^3 te^{-st}dt = \left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^3 = -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}$$

The second inegral is easier

$$\int_3^\infty 3e^{-st}dt = \left(\frac{3e^{-st}}{-s}\right]_3^\infty = \frac{3e^{-3s}}{s}$$

Putting it all together

$$L(f(t)) = -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{3e^{-3s}}{s} = -\frac{e^{-3s}}{s^2} + \frac{1}{s^2}$$

Question 2 continued on next page.

Q2. (b) Use Laplace Transforms to solve the following initial value problem.

$$\ddot{y} + 9y = f(t) = \left\{ \begin{array}{cc} t & t < 3 \\ 3 & 3 \le t \end{array} \right] \qquad \text{with } y(0) = 1 \text{ and } \dot{y}(0) = 0$$

Solution

Let $L(y(t)) = Y(s)$ and use the results for derivatives:

$$L(\dot{y}(t)) = sY(s) - y(0) \quad \text{and} \quad L(\ddot{y}(t)) = s^2Y(s) - sy(0) - \dot{y}(0).$$

Then take Laplace Transforms of the whole equation

$$s^2Y(s) - sy(0) - \dot{y}(0) + 9Y(s) = \frac{1 - e^{-3s}}{s^2}$$

Using the given initial conditions and solving for $Y(s) = \dfrac{s}{s^2+9} + \dfrac{1-e^{-3s}}{s^2(s^2+9)}$ The inverse laplace transform of the first term is straightforward. From the formula sheet $L(\cos(3t)) = \dfrac{s}{s^2+9}$. But to take the inverse laplace transform of the second term requires us to first use partial fractions on $\dfrac{1}{s^2(s^2+9)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{(s^2+9)}$.

$$\text{This implies that} \quad 1 = As(s^2+9) + B(s^2+9) + (Cs+D)s^2.$$

Equating coefficients of powers of s is proves a good way to solve for A, B, C, D here but you can also substitute values for s.

Equating the coefficients of s^3	$0 = A + C \Rightarrow C = -A$
Equating the coefficients of s^2	$0 = B + D \Rightarrow D = -B$
Equating the coefficients of s	$0 = 9A \Rightarrow A = 0 \Rightarrow C = 0$
Equating the constants	$1 = 9B \Rightarrow B = \frac{1}{9},\ D = -\frac{1}{9}$

So finally $\dfrac{1}{s^2(s^2+9)} = \dfrac{1}{9s^2} - \dfrac{1}{9(s^2+9)}$ and $L^{-1}\left(\dfrac{1}{s^2(s^2+9)}\right) = \dfrac{1}{27}(3t - \sin(3t))$

Now use the second shift theorem

$$L^{-1}\left(\frac{e^{-3s}}{s^2(s^2+9)}\right) = \frac{1}{27}[3(t-3) - \sin(3(t-3))]u(t-3)$$

Putting it all together, the inverse transform of $Y(s)$ and the solution to the intial value problem is

$$y(t) = \cos(3t) + \frac{1}{27}\left(3t - \sin(3t) - [3(t-3) - \sin(3(t-3))]u(t-3)\right).$$

(24 marks)

Question 2 continued on next page. TURN OVER

Table of Laplace Transforms

$f(t)$	$F(s)$
K	$\frac{K}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$e^{at}f(t)$	$F(s-a)$
$f(t-k)u(t-k) = \left\{ \begin{array}{cc} 0 & t < k \\ f(t-k) & k \leq t \end{array} \right]$	$e^{-ks}F(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{n-1}(0)$