INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

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First Semester Examination, June, 2005

MATH2010

ANALYSIS OF ORDINARY DIFFERENTIAL EQUATIONS

(Unit Courses, Inf. Tech.)

Time: ONE HOUR for working

Ten minutes for perusal before examination begins

Check that this examination paper has 10 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!
Students should attempt **all** questions.
The exam paper is a total of 65 marks
The marks allocated to each part of each question are as indicated.
Calculators allowed, but all memory must be cleared beforehand.
Check that this examination paper has 10 printed pages!

FAMILY NAME (PRINT): ____________________________

GIVEN NAMES (PRINT): ____________________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ____________________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		2	
TOTAL MARKS			

Q1. (a) **Sketch the trajectories of the following system in the phase plane**, indicating the direction of flow, and **classify the type and stability** of the fixed point at the origin.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

State the equation of any straight line trajectories.
Find the slope of the flow on the y_1 axis and on the y_2 axis and indicate how you have used your results on your sketch. **(17 marks)**

Solution You can work out the type and stability of the critical point at the origin by **either** calculating the determinant, trace and $trace^2 - 4det$ of the matrix **or** by calculating the eigenvalues of the matrix.

Here the determinant, trace and $trace^2 - 4det$ of the matrix are 8, 6 and $36 - 4 \times 8 = 4 >$ which implies that the critical point is an **UNstable node.**

Alternatively the eigenvalues from the matrix are given by $\lambda^2 - 6\lambda + 8 = 0$ that is $\lambda_1 = 2$ and $\lambda_2 = 4$, which also implies that the critical point is an **UNstable node.**

To sketch an node you first need to find the straightline solutions which are given by the eigen vectors of the matrix. Here

$$\lambda_1 = 2 \quad \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \lambda_2 = 4 \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So the straightline solutions are

$$y_2 = 3y_1 \text{ from } \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ on which } y_1 = c_1 e^{2t} \Rightarrow \text{exponential growth}$$

$$y_2 = y_1 \text{ from } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ on which } y_1 = c_1 e^{4t} \Rightarrow \text{exponential growth}$$

For exponential growth the direction of flow is away from the critical point, as shown in the diagram.

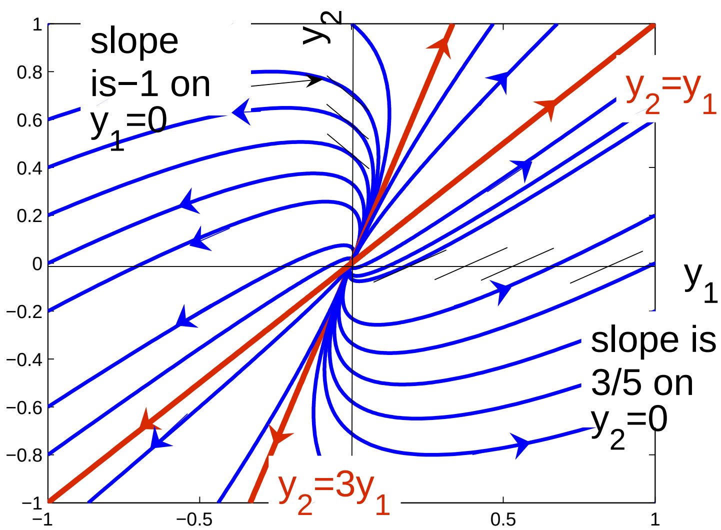
Question 1 continued on next page

By the chain rule the slope of the trajectories is given by

$$\frac{dy_2}{dy_1} = \frac{\dot{y_2}}{\dot{y_1}} = \frac{3y_1 + y_2}{5y_1 - y_2}$$

So that on $y_2 = 0$ the slope is $\dfrac{dy_2}{dy_1} = \dfrac{3}{5}$ and
on $y_1 = 0$ the slope is $\dfrac{dy_2}{dy_1} = -1$.

Since $|2| < |4|$ the trajectories emerge from the origin tangent to the solution on $y_2 = 3y_1$.



Question 1 continued on next page.

Q1. (b) Find **all** the **critical points** for the following nonlinear system.

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} -4r + 3r^2 - 2rs \\ s - rs + s^2 \end{pmatrix}$$

Then use Linearization to find the type and stability of the critical points that lie in the first quadrant, i.e. for which $r \geq 0$ AND $s \geq 0$.

Solution

The critical points are given by $\dot{r} = 0$ AND $\dot{s} = 0$. Here this means that

$$-4r + 3r^2 - 2rs = 0 \quad \text{and} \quad s - rs + s^2 = 0$$

$$\Rightarrow \{r = 0 \quad \text{or} \quad -4 + 3r - 2s = 0\} \quad \text{and} \quad \{s = 0 \quad \text{or} \quad 1 - r + s = 0\}$$

Potentially there are 4 possible cases:
$r = 0$ and $s = 0$, which gives $(0,\ 0)$.
$r = 0$ and $1 - r + s = 0$, which gives $(0,\ -1)$.
$-4 + 3r - 2s = 0$ and $s = 0$, which gives $(\frac{4}{3},\ 0)$ and
$-4 + 3r - 2s = 0$ and $1 - r + s = 0$, which gives $(2,\ 1)$.

So there are **four critical points**: $(0,\ 0)$, $(0,\ -1)$, $(\frac{4}{3},\ 0)$ and $(2,\ 1)$, three of which lie in the first quadrant: $(0,\ 0)$, $(\frac{4}{3},\ 0)$ and $(2,\ 1)$.

The linearized matrix is

$$\mathbf{Df} = \begin{pmatrix} -4 + 6r - 2s & -2r \\ -s & 1 - r + 2s \end{pmatrix}$$

Now $\mathbf{Df}(\mathbf{0},\ \mathbf{0}) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$ which has determinant equal to -4 implying that the critical point is a **saddle**, which is **UNstable**. (Alternatively the eigenvalues are -4 and 1 which are real and one is negative while the other is positive which implies that the critical point is a **saddle**).

$\mathbf{Df}(\frac{\mathbf{4}}{\mathbf{3}},\ \mathbf{0}) = \begin{pmatrix} 4 & -\frac{8}{3} \\ 0 & -\frac{1}{3} \end{pmatrix}$ which has determinant equal to $-\frac{4}{3} < 0$ implying that the critical point is a **saddle**, which is **UNstable**. (Alternatively the eigenvalues are 4 and $-\frac{1}{3}$ which implies that the critical point is a **saddle**).

$\mathbf{Df}(\mathbf{2},\ \mathbf{1}) = \begin{pmatrix} 6 & -4 \\ -1 & 1 \end{pmatrix}$ which has determinant equal to 2 and trace equal to 7, also $trace^2 - 4det > 0$ so that the critical point is an **Unstable node**. (Alternatively the eigenvalues are given by $\lambda^2 - 7\lambda + 2 = 0$ which has real positive solutions $\lambda_{\pm} = (7 \pm \sqrt{41})/2$ so that the critical point is an **Unstable node**.)

(16 marks)

Question 2 on next page. TURN OVER

Q2. (a) Find the **inverse** Laplace Transform of

$$\frac{e^{-4s}}{((s-3)^2+9)(s-3)^2}$$

State clearly any theorems that you use.

Solution

Method 1

Use partial fractions:

$$\text{Let} \quad \frac{1}{((s-3)^2+9)(s-3)^2} = \frac{A}{(s-3)} + \frac{B}{(s-3)^2} + \frac{Cs+D}{((s-3)^2+9)}$$

This implies that $1 = A(s-3)((s-3)^2+9) + B((s-3)^2+9) + (Cs+D)(s-3)^2$. Here the best method is a combination of taking values for s and equating coefficents of the powers of s.

Equating the coefficents of s^3 gives $0 = A + C \Rightarrow C = -A$ Setting $s = 3$ gives $1 = 9B \Rightarrow B = \frac{1}{9}$. Equating the coefficents of s^2 gives $0 = -9A+B-6C+D \Rightarrow 3A = D + \frac{1}{9}$ Setting $s = 2$ gives $1 = -10A + 10B + 2C + D$ or $12A = \frac{1}{9} + D$. Given the previous result this implies that $A = 0 \Rightarrow C = 0$. Then $B = \frac{1}{9}$ and $D = -\frac{1}{9}$.

$$\text{So that} \quad \frac{1}{((s-3)^2+9)(s-3)^2} = \frac{1}{9(s-3)^2} + \frac{1}{9((s-3)^2+9)}$$

Now $L^{-1}\left(\dfrac{1}{(s-3)^2}\right) = te^{3t}$. Also $L^{-1}\left(\dfrac{1}{((s-3)^2+9)}\right) = \dfrac{1}{3}e^{3t}\sin(3t)$. Further using the second shifting theorem:

$$L^{-1}\left(\frac{e^{-4s}}{((s-3)^2+9)(s-3)^2}\right) = \frac{e^{3(t-4)}}{9}\left((t-4) - \frac{1}{3}\sin(3(t-4))\right)u(t-4)$$

Method 2

Use convolution:

$$L^{-1}\left(\frac{1}{((s-3)^2+9)(s-3)^2}\right) = \int_0^t \tau e^{3\tau}\frac{1}{3}e^{3(t-\tau)}\sin(3(t-\tau))d\tau = \frac{1}{3}e^{3t}\int_0^t \tau\sin(3(t-\tau))d\tau$$

Now

$$\int_0^t \tau\sin(3(t-\tau))d\tau = \left(\frac{\tau\cos(3(t-\tau))}{3} + \frac{\sin(3(t-\tau))}{9}\right]_0^t = \frac{t}{3} - \frac{\sin(3t)}{9}$$

So that

$$L^{-1}\left(\frac{1}{((s-3)^2+9)(s-3)^2}\right) = \frac{e^{3t}}{27}(3t - \sin(3t))$$

Now use the second shifting theorem as in the previous method to get the final result.

(12 marks)

Question 2 continued on next page.

Q2. (b) Use Laplace Transforms to solve the following **system of equations** with the given initial values.

$$\dot{y_1} = 2y_1 + 3y_2 + f(t)$$

$$\dot{y_2} = 2y_1 + y_2.$$

Where $f(t) = \left\{ \begin{array}{ll} 5 & t < 2 \\ 0 & 2 \leq t \end{array} \right]$ and $y_1(0) = 0$ and $y_2(0) = 0$.

Solution

Let $Y_i(s) = L(y_i(t))$ and take laplace transforms of both equations:

$$sY_1(s) - y_1(0) = 2Y_1(s) + 3Y_2(s) + F(s)$$

$$sY_2(s) - y_2(0) = 2Y_1(s) + Y_2(s)$$

Where $F(s) = L(f(t)) = L(5 - 5u(t-2)) = \dfrac{5(1-e^{2s})}{s}$.

Given that $y_1(0) = 0$ and $y_2(0) = 0$ this becomes

$$(s-2)Y_1(s) = 3Y_2(s) + F(s) \quad \text{and} \quad (s-1)Y_2(s) = 2Y_1(s)$$

$$\Rightarrow (s-1)(s-2)Y_1(s) = 6Y_1(s) + (s-1)F(s)$$

$$Y_1(s) = \frac{(s-1)F(s)}{(s-1)(s-2)-6} = \frac{(s-1)F(s)}{(s-4)(s+1)} \qquad Y_2(s) = \frac{2Y_1(s)}{(s-1)} = \frac{2F(s)}{(s-4)(s+1)}$$

Now using the fact that $F(s) = \dfrac{5(1-e^{2s})}{s}$ gives

$$Y_1(s) = \frac{5(s-1)(1-e^{2s})}{s(s-4)(s+1)} \qquad Y_2(s) = \frac{10(1-e^{2s})}{s(s-4)(s+1)}$$

To find the inverse transform use Partial Fractions, for $Y_1(s)$:

$$\text{Let} \quad \frac{5(s-1)}{s(s-4)(s+1)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s-4)}$$

This implies that $5(s-1) = A(s+1)(s-4) + Bs(s-4) + Cs(s+1)$.
Here to solve for A, B, C substitute $s = 0, -1$ and 4:
If $s = 0$ then $-5 = -4A \Rightarrow A = \frac{5}{4}$, if $s = -1$ then $-10 = 5B \Rightarrow B = -2$ and if $s = 4$ then $15 = 20C \Rightarrow C = \frac{3}{4}$.

$$\frac{5(s-1))}{s(s-4)(s+1)} = \frac{5}{4s} - \frac{2}{(s+1)} + \frac{3}{4(s-4)} \quad \text{and} \quad L^{-1}\left(\frac{5(s-1))}{s(s-4)(s+1)}\right) = \frac{5}{4} - 2e^{-t} + \frac{3}{4}e^{4t}$$

Then use the second shifting theorem to find $y_1(t)$:

$$y_1(t) = \frac{5}{4} - 2e^{-t} + \frac{3}{4}e^{4t} - \left(\frac{5}{4} - 2e^{-(t-2)} + \frac{3}{4}e^{4(t-2)}\right) u(t-2)$$

Question 2 continued on next page. TURN OVER

Now for $y_2(t) = L^{-1}(Y_2(s))$ also use Partial Fractions:

$$\text{Let} \quad \frac{10}{s(s-4)(s+1)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s-4)}$$

This implies that $10 = A(s+1)(s-4) + Bs(s-4) + Cs(s+1)$.
Here to solve for A, B, C to substitute $s = 0, -1$ and 4:
If $s = 0$ then $10 = -4A \Rightarrow A = -\frac{5}{2}$, if $s = -1$ then $10 = 5B \Rightarrow B = 2$ and if $s = 4$ then $10 = 20C \Rightarrow C = \frac{1}{2}$.

$$\frac{10}{s(s-4)(s+1)} = -\frac{5}{2s} + \frac{2}{(s+1)} + \frac{1}{2(s-4)} \quad \text{and} \quad L^{-1}\left(\frac{10}{s(s-4)(s+1)}\right) = -\frac{5}{2} + 2e^{-t} + \frac{1}{2}e^{4t}$$

Then use the second shifting theorem to find $y_2(t)$:

$$y_2(t) = -\frac{5}{2} + 2e^{-t} + \frac{1}{2}e^{4t} - \left(-\frac{5}{2} + 2e^{-(t-2)} + \frac{1}{2}e^{4(t-2)}\right) u(t-2)$$

(20 marks)

Table of Laplace Transforms

$f(t)$	$F(s)$
K	$\frac{K}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$e^{at}f(t)$	$F(s-a)$
$f(t-k)u(t-k) = \left\{ \begin{array}{cc} 0 & t < k \\ f(t-k) & k \le t \end{array} \right]$	$e^{-ks}F(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{n-1}(0)$