INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Second Semester Examination, November, 2004

MATH2100

ADVANCED MATHEMATICAL ANALYSIS

(Unit Courses, Inf. Tech.)

Time: **ONE HOUR** for working

Ten minutes for perusal before examination begins

Check that this examination paper has 18 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!
Students should attempt **all** questions.
The exam paper is a total of 65 marks
The marks allocated to each part of each question are as indicated.
Calculators allowed, but all memory must be cleared beforehand.
Check that this examination paper has 18 printed pages!

FAMILY NAME (PRINT): ____________________

GIVEN NAMES (PRINT): ____________________

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: ____________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

Q1. (a) **Sketch the trajectories of the following system in the phase plane**, indicating the direction of flow, and **classify the type and stability** of the fixed point at the origin.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

State the equation of any straight line trajectories.
State (briefly) why the direction of flow is as you have indicated.

(8.5 marks)

Question 1 continued on next page.

Q 1 (a). **Working space only**

Question 1 continued on next page.

Q1. (b) Find all the **critical points** for the following nonlinear system.

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 3r - rs - r^2 \\ -s + 2rs \end{pmatrix}$$

Then use Linearization to find the type and stability of the critical point **that does not** lie on an axis, i.e. for which $r \neq 0$ AND $s \neq 0$.

(8 marks)

Question 1 continued on next page.

Q 1 (b). **Working space only**

Q2. (a) Prove that the Laplace Transform of

$$f(t) = \left\{ \begin{array}{ll} t & t < 3 \\ 3 & 3 \leq t \end{array} \right] \quad \text{is} \quad F(s) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$

State clearly any theorems that you use. **(4 marks)**

Q2. (b) Use Laplace Transforms to solve the following initial value problem.

$$\ddot{y} + 9y = f(t) = \left\{ \begin{array}{ll} t & t < 3 \\ 3 & 3 \le t \end{array} \right] \qquad \text{with } y(0) = 1 \text{ and } \dot{y}(0) = 0$$

(12 marks)

Question 2 continued on next page.

Q 2 (b). **Working space only**

Q 3 (a). Show that the Fourier Series corresponding to the function defined by

$$f(x) = \alpha\,, \qquad -L < x < 0\,; \qquad f(x) = \beta\,, \qquad 0 < x < L\,;$$
$$\text{and} \qquad f(x + 2L) = f(x)\,, \qquad -\infty < x < \infty\,,$$

where α and β are constants, is given by

$$\frac{1}{2}(\alpha + \beta) - \frac{2(\alpha - \beta)}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)} \sin\left[\frac{(2m - 1)\pi x}{L}\right]\,. \qquad (\star)$$

(8.5 marks)

Question 3 continued on next page.

Q 3 (a). **Working space only**

Question 3 continued on next page.

Q 3 (b). Briefly explain the value taken by the series ($\star$) at $x = 0$.

(2.5 marks)

Q 3 (c). By considering the value to which the series ($\star$) must converge at $x = L/2$ deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}\,.$$

(3 marks)

Question 3 continued on next page.

Q 3 (d). Describe and explain in a few words what happens to the series ($\star$) when (a) $\alpha = \beta$ and (b) $\alpha = -\beta$.

(2.5 marks)

Q 4 (a). You are given (**no need to check!**) that the function $G(x-y,t)$ defined by

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-(x-y)^2/(4c^2t)}$$

satisfies

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

Show that $u(x,t)$ defined by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

satisfies the 1-dimensional Heat Equation for $-\infty < x < \infty$ and $t > 0$, and also the initial condition

$$\lim_{t\to 0_+} u(x,t) = f(x)\,, \qquad -\infty < x < \infty\,.$$

(4.5 marks)

Question 4 continued on next page. TURN OVER

Q 4 (a). **Working space only**

Q 4 (b). A very long cylindrical iron bar, with unknown thermal diffusivity c^2, lies along the positive x-axis . The sides of the bar and the end at $x = 0$ are thermally insulated, and the temperature u inside the bar is a function only of x and of time t. The temperature distribution in the bar at $t = 0$ is given by

$$u(x,0) = F(x) = \begin{cases} u_0 \text{ (const.)} & , \quad 0 < x < L \\ 0 & , \quad L < x < \infty \end{cases} .$$

Deduce that

$$u(x,t) = \frac{1}{2} u_0 \left\{ \operatorname{erf}\left(\frac{\mathrm{x}+\mathrm{L}}{\sqrt{4\mathrm{c}^2\mathrm{t}}} \right) - \operatorname{erf}\left(\frac{\mathrm{x}-\mathrm{L}}{\sqrt{4\mathrm{c}^2\mathrm{t}}} \right) \right\}$$

for $x > 0$, $t > 0$, where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv .$$

(9 marks)

Question 4 continued on next page.

Q 4 (b). **Working space only**

Question 4 continued on next page.

Q 4 (c). Given that erf(0.5)$\approx$ 0.5, deduce that the thermal diffusivity c^2 is given approximately by

$$c^2 = L^2/t_1 \,,$$

where t_1 is the time taken for the temperature at the face $x = 0$ to reach the value $u_0/2$.

(2.5 marks)

Table of laplace Transforms on next page.

Table of Laplace Transforms

$f(t)$	$F(s)$
K	$\dfrac{K}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\dfrac{\alpha}{s^2+\alpha^2}$
$e^{at}f(t)$	$F(s-a)$
$f(t-k)u(t-k) = \left\{ \begin{array}{cc} 0 & t<k \\ f(t-k) & k \le t \end{array} \right]$	$e^{-ks}F(s)$
$\displaystyle\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{n-1}(0)$