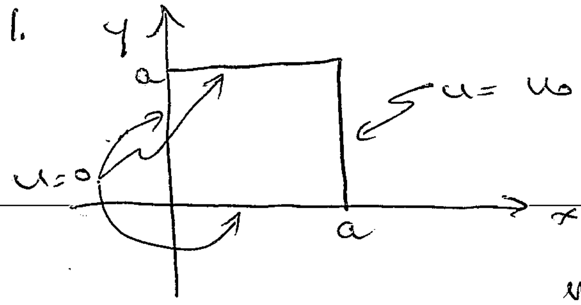
MATH 2100 Solutions 10 (= MATH2011 Solutions 5) AJB

10-1

1.



As shown in lectures (with $x \leftrightarrow y$)

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\frac{n\pi x}{a}\right) \quad ②$$

where
$$B_n = \frac{2}{a\sinh(n\pi)} \int_0^a u_0 \sin\left(\frac{n\pi y}{a}\right) dy \quad ②$$

$$= \frac{-2u_0}{n\pi\sinh(n\pi)} \left[\cos\left(\frac{n\pi y}{a}\right)\right]_0^a$$

$$= \frac{2u_0}{n\pi\sinh(n\pi)} \left[1-(-1)^n\right]$$

$$= \frac{2u_0}{\pi} \begin{cases} 0 & n = 2m \text{ even} \\ \dfrac{2}{(2m-1)\sinh[(2m-1)\pi]} & n = 2m-1 \text{ odd} \quad m = 1,2,\dots \end{cases} \quad ②$$

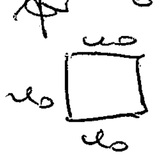
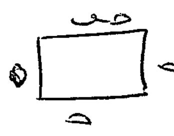
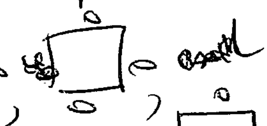
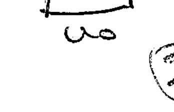
So
$$u(x,y) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin\left[\frac{(2m-1)\pi y}{a}\right] \sinh\left[\frac{(2m-1)\pi x}{a}\right]}{(2m-1)\sinh[(2m-1)\pi]} \quad ①$$

Then
$$u\left(\tfrac{a}{2},\tfrac{a}{2}\right) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin\left[(2m-1)\frac{\pi}{2}\right] \sinh\left[\frac{(2m-1)\pi}{2}\right]}{(2m-1)\sinh[(2m-1)\pi]}$$

Now $\sin\left[\frac{(2m-1)\pi}{2}\right] = (-1)^{m+1}$ and $\sinh[(2m-1)\pi] = 2\sinh\left[\frac{(2m-1)\pi}{2}\right]\cosh\left[\frac{(2m-1)\pi}{2}\right]$

So
$$u\left(\tfrac{a}{2},\tfrac{a}{2}\right) = \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)\cosh[(2m-1)\pi/2]}. \quad \circledast \quad ②$$

If all four faces are kept at u_0, obviously $u(x,y) = u_0$ for $0<x<a$ and $0<y<a$. But the solution for

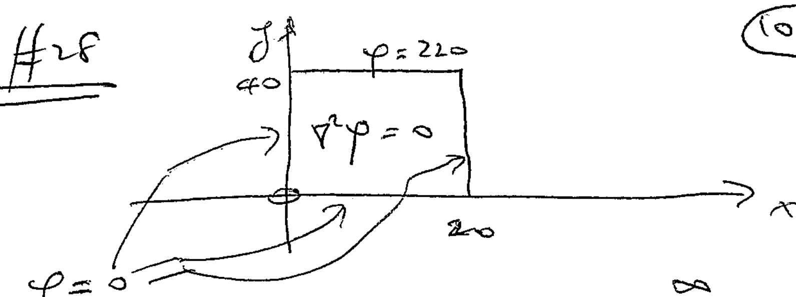
 = sum of solns for , , , 

and obviously each of the four on the right gives same value for $u\left(\frac{a}{2},\frac{a}{2}\right)$ as in $\circledast$. Therefore ②

$$\frac{8u_0}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)\cosh[(2m-1)\pi/2]} = u_0$$, & hence result.

(10.3)

K12.5 #28



This is much like Qu 1, above

From lecture notes:-

$$u(x,y)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

$a = 20,\ b = 40$

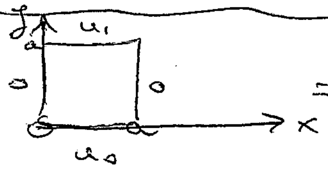
$\varphi_0 = 220$

$$B_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a \varphi_0 \sin\left(\frac{n\pi x}{a}\right)dx$$

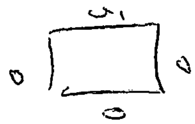
$$=\begin{cases} 0 & n = 2m \text{ even} \\ \dfrac{4\varphi_0}{(2m-1)\pi\sinh\left(\frac{(2m-1)\pi b}{a}\right)} & n = 2m-1 \text{ odd}\end{cases}$$

So
$$u(x,y) = \frac{4\varphi_0}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}{(2m-1)\sinh\left[(2m-1)\pi b/a\right]}$$

K12.5 #32

Solⁿ for 

= Solⁿ for 

+ solⁿ for 

Solⁿ for first on right is as in Qu. 1 above, with $y \to a - y$.

$$= \frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left[\frac{(2m-1)\pi x}{a}\right]\sinh\left[\frac{(2m-1)\pi(a-y)}{a}\right]}{(2m-1)\sinh\left[(2m-1)\pi\right]}$$

Solⁿ for second on right is as in Qu 1 above, with $u_0 \to u_1$

$$= \frac{4u_1}{\pi}\sum_{m=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sinh\left[\frac{(2m-1)\pi y}{a}\right]}{(2m-1)\sinh\left[(2m-1)\pi\right]}$$

So solⁿ for given situation is

$$u(x,y) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}}{(2m-1)\sinh\left[(2m-1)\pi\right]}\left\{u_0\sinh\left[\frac{(2m-1)\pi(a-y)}{a}\right] + u_1\sinh\left[\frac{(2m-1)\pi y}{a}\right]\right\}$$

2. $u(x,y) = x^2 - y^2 + e^{+x^2} e^{-(y+1)^2} \cos[2x(y+1)]$ (10.2)

$\Rightarrow u_x = 2x + 2x e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]$ (1)

$u_y = -2y - 2(y+1) e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2x e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]$ (1)

$u_{xx} = 2 + 2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] + 4x^2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)]$
(1) $- 4x(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] - 4(y+1)^2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)]$.

$u_{yy} = -2 - 2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] + 4(y+1)^2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)]$
(1) $+ 4x(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] - 4x^2 e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)]$

So $u_{xx} + u_{yy} = 0$ So u is harmonic. (1)

To find $v(x,y)$:–

$v_y = u_x = 2x + 2x e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]$

$\Downarrow$

$v = 2xy + e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] + H(x)$

$\Downarrow$

$\Rightarrow v_x = 2y + 2x e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] + 2(y+1) e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] + H'(x)$

$= -u_y + H'(x)$

So $v_x = -u_y \Rightarrow H'(x) = 0 \Rightarrow H(x) = \text{const.}$

So $v = 2xy + e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] + \text{const.}$ (2)

$\nabla u = \{2x + 2x e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]\} \underline{i}$
$+ \{-2y - 2(y+1) e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2x e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]\} \underline{j}$

$\nabla v = \{2y + 2x e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)] + 2(y+1) e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)]\} \underline{i}$
$+ \{2x + 2x e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]\} \underline{j}$

$\nabla u \cdot \nabla v = (A\underline{i} + B\underline{j}) \cdot (-B\underline{i} + A\underline{j}) = -AB + AB = 0$.

$A = 2x + 2x e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2(y+1) e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]$

$B = -2y - 2(y+1) e^{x^2} e^{-(y+1)^2} \cos[2x(y+1)] - 2x e^{x^2} e^{-(y+1)^2} \sin[2x(y+1)]$.

(2)