INTERNAL STUDENTS ONLY THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Zeroth Semester Examination, Ides March, 709 AUC

MATH2100

ADVANCED MATHEMATICAL ANALYSIS

(Unit Courses, Inf. Tech.)

Time: TWO Hours for working

Ten minutes for perusal before examination begins

Check that this examination paper has ~~28~~ 29 printed pages!

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION PAPER!

Answer as many questions as time permits, and make sure to answer AT LEAST ONE of Questions A1, A2, A3 and AT LEAST ONE of Questions B1, B2, B3.
It is expected that AT LEAST THREE questions IN TOTAL will be attempted.
All questions carry the same number of marks.
Pocket calculators without ASCII capabilities may be used.

TRIAL EXAMINATION

FAMILY NAME (PRINT): SHOES

GIVEN NAMES (PRINT): JIM

STUDENT NUMBER: 0 1 2 3 4 5 6 7

SIGNATURE: J. Shoes

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
A1		B1	
A2		B2	
A3		B3	
TOTAL MARKS			

A1

Q1. (a) Find the general solution of the system

$$\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t), \quad A = \begin{bmatrix} -1 & -1 \\ 5 & 1 \end{bmatrix}.$$

Find e'values & e'vectors of A:-

E'value condition: $\begin{vmatrix} -1-\lambda & -1 \\ 5 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow \quad (\lambda+1)(\lambda-1)+5 = 0$$

$$\Rightarrow \quad \lambda^2+4 = 0$$

$$\Rightarrow \quad \lambda = \pm 2i$$

E' vectors:-

$\underline{\lambda = 2i}$ $\quad \begin{bmatrix} -1-2i & -1 \\ 5 & 1-2i \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad \left.\begin{array}{l} (-1-2i)u - v = 0 \\ 5u + (1-2i)v = 0 \end{array}\right\} \Rightarrow v = -(1+2i)u$$

$\underline{\text{Note}}$: These two equations are the $\underline{\text{same}}$.

Choose $u=1$ so e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -(1+2i) \end{pmatrix}$

$\underline{\lambda = -2i}$ - Similar calculation — or see ~~that~~ just replace i by $-i$ throughout, so

$$\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -(1-2i) \end{pmatrix}$$

Question 1(a) continued on next page.

A

Q1. (a) **Working space only**

Now $\underset{\sim}{x}^{(1)} e^{2it}$ and $\underset{\sim}{x}^{(2)} e^{-2it}$ are solutions (linearly independent). So general solution is

$$\underset{\sim}{y}(t) = A\,\underset{\sim}{x}^{(1)} e^{2it} + B\,\underset{\sim}{x}^{(2)} e^{-2it}$$

$$= A\begin{pmatrix} 1 \\ -(1+2i) \end{pmatrix} e^{2it} + B\begin{pmatrix} 1 \\ -(1-2i) \end{pmatrix} e^{-2it}$$

Question 1 continued on next page.

A1

Q1. (b) Write the general solution in real form.

We have $\underset{\sim}{y} = A\begin{pmatrix}1\\-(1+2i)\end{pmatrix}(\cos 2t + i\sin 2t)$

$$+ B\begin{pmatrix}1\\-(1-2i)\end{pmatrix}(\cos 2t - i\sin 2t)$$

$$= \begin{pmatrix}(A+B)\cos 2t + i(A-B)\sin 2t\\ (A+B)[-\cos 2t + 2\sin 2t] + i(A-B)[-2\cos 2t - \sin 2t]\end{pmatrix}$$

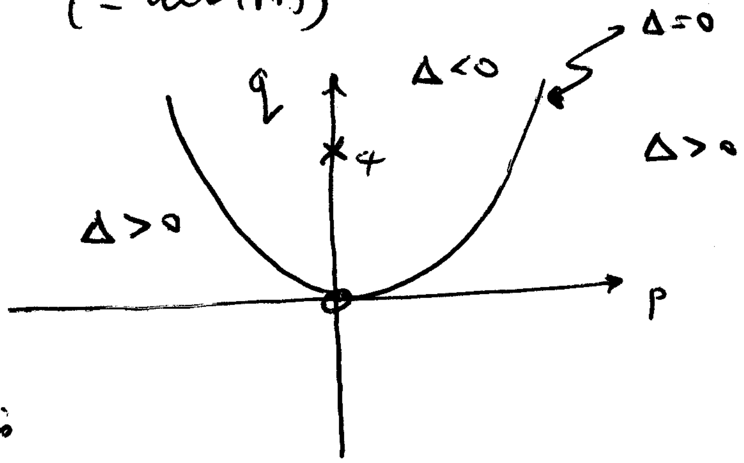
$$= C\begin{pmatrix}\cos 2t\\ -\cos 2t + 2\sin 2t\end{pmatrix} + D\begin{pmatrix}\sin 2t\\ -2\cos 2t - \sin 2t\end{pmatrix}$$

(c) Identify the type and stability of the equilibrium (critical) point at the origin.

$p = -1 + 1 = 0 \quad (= \mathrm{tr}(A))$

$q = (-1)1 - (-1)5 = 4 \quad (= \det(A))$

$\Delta = p^2 - 4q = -16$



Equilibrium point is a centre & hence is stable (but not stable & attractive)

Question 1 continued on next page.

A1

1. (d) Determine the slopes of trajectories where they cross the lines $y_2 = y_1$, $y_2 = -y_1$, $y_2 = -5y_1$ and $y_1 = 0$. Determine the directions of trajectories where they cross the line $y_1 = 0$. Use your results to help sketch some trajectories.

Slopes: $\dfrac{dy_2}{dy_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{5y_1 + y_2}{-y_1 - y_2}$

when $y_2 = y_1$: slope $= \dfrac{6y_1}{-2y_1} = -3$

when $y_2 = -y_1$: slope $= \dfrac{4y_1}{0} = \pm\infty$

when $y_2 = -5y_1$: slope $= \dfrac{0}{4y_1} = 0$

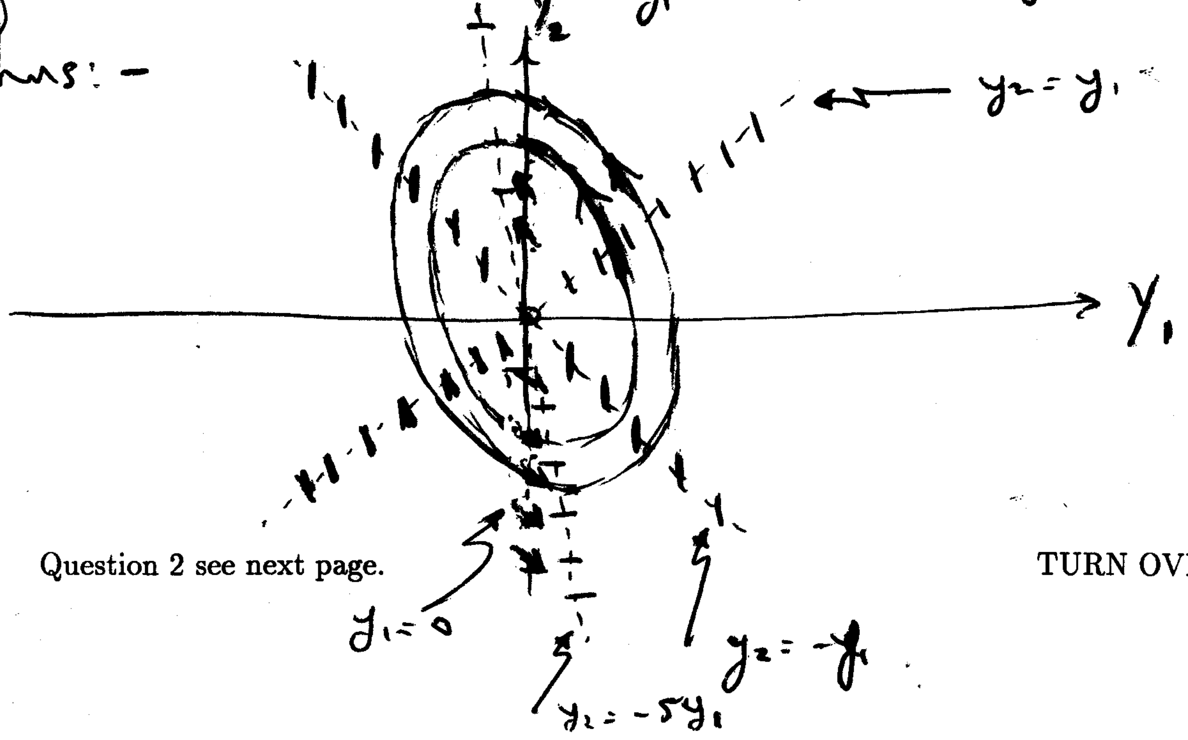
when $y_1 = 0$: slope $= \dfrac{y_2}{-y_2} = -1$

Directions: $\dfrac{dy_1}{dt} = -y_1 - y_2$

so when $y_1 = 0$, $\dfrac{dy_1}{dt} = -y_2$

– so y_1 incr. when $y_2 < 0$
y_1 decr. when $y_2 > 0$.

Thus:–



A2

Q2. (a) Rewrite

$$y''(t) - 2y\,(t) + y(t)^2 + y(t)^3 = 0$$

as a system of two coupled ordinary differential equations, and locate the equilibrium points of the system.

Let $y_1 = y$, $y_2 = y'$

So $y_1' = y' = y_2$

$y_2' = y'' = 2y - y^2 - y^3 = 2y_1 - y_1^2 - y_1^3$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= 2y_1 - y_1^2 - y_1^3 \end{aligned}} \begin{aligned} &= F_1(y_1, y_2) \\ &= F_2(y_1, y_2) \end{aligned}$$

Equilibrium points are where

$y_1' = 0 = y_2'$

So where $y_2 = 0$ and $2y_1 - y_1^2 - y_1^3 = 0$

$-y_1(y_1^2 + y_1 - 2) = 0$

$-y_1(y_1 + 2)(y_1 - 1) = 0$

$y_1 = 0, 1, -2.$

So equilibrium points are

$(0,0), (1,0), (-2,0)$

Question 2 continued on next page.

(A2) Q2. (b) Determine the type and stability of the equilibrium points by linearization.

$$F_{11} = \frac{\partial F_1}{\partial y_1} = 0, \quad F_{12} = \frac{\partial F_1}{\partial y_2} = 1$$

$$F_{21} = \frac{\partial F_2}{\partial y_1} = 2 - 2y_1 - 3y_1^2, \quad F_{22} = \frac{\partial F_2}{\partial y_2} = 0$$

<u>At (0,0):</u> $\underset{\sim}{y}' \approx F\underset{\sim}{y}$, $F = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$p = 0$, $q = -2$
$\Delta = 8$ saddle pt. unstable.

<u>At (1,0):</u> $\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$ $\begin{pmatrix} Y_1 = y_1 - 1 \\ Y_2 = y_2 \end{pmatrix}$ $F = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$. $p = 0$, $q = 3$, $\Delta = -12$ centre stable

At $(-2, 0)$: $F = \begin{bmatrix} 0 & 1 \\ -6 & 0 \end{bmatrix}$ $p = 0$, $q = 6$, $\Delta = -24$

$\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$ $\begin{pmatrix} Y_1 = y_1 + 2 \\ Y_2 = y_2 \end{pmatrix}$

centre stable

Question 2 continued on next page.

A2

Q2. (c) Use the Second Shifting Theorem (see Table) to find the Inverse Laplace Transform of

$$F(s) = \frac{e^{-3s}}{s^3}.$$

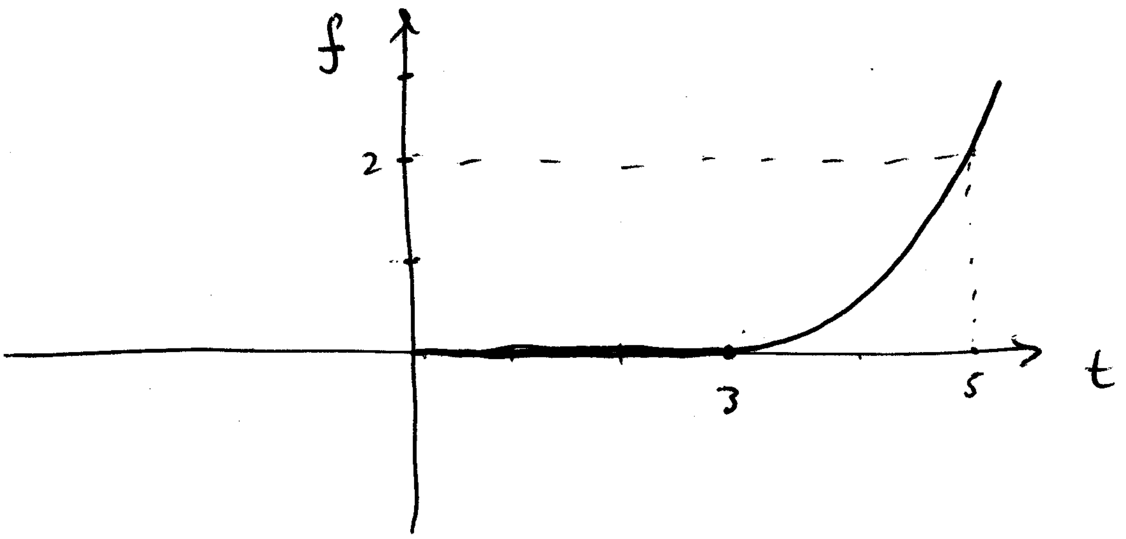
Sketch the resulting inverse function, $f(t)$.

Since $\frac{1}{s^3}$ has Inverse L.T. $\frac{1}{2}t^2$

we have

$$f(t) = \begin{cases} 0 & t < 3 \\ \frac{1}{2}(t-3)^2 & t > 3 \end{cases}$$

$$= \frac{1}{2}(t-3)^2 u(t-3)$$



A3

Q3. (a) Using the Table provided,

(i) Representing the hyperbolic functions in terms of exponential functions, and applying the First Shifting Theorem (see Table), show that

$$\mathcal{L}(\cosh t \sin t) = \frac{s^2+2}{s^4+4}.$$

$\cosh t = \frac{1}{2}(e^t + e^{-t})$

$\cosh t \sin t = \frac{1}{2} e^t \sin t + \frac{1}{2} e^{-t} \sin t$

$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$

$\Rightarrow \mathcal{L}(e^t \sin t) = \frac{1}{(s-1)^2+1}$

and $\mathcal{L}(e^{-t} \sin t) = \frac{1}{(s+1)^2+1}$

So $\mathcal{L}(\cosh t \sin t) = \frac{1}{2}\left\{ \frac{1}{(s-1)^2+1} + \frac{1}{(s+1)^2+1} \right\}$

$$= \frac{1}{2} \frac{(s-1)^2+1 + (s+1)^2+1}{[(s-1)^2+1][(s+1)^2+1]}$$

$$= \frac{1}{2} \frac{2s^2+4}{(s^2-2s+2)(s^2+2s+2)}$$

$$= \frac{s^2+2}{s^4+4}$$

Question 3 continued on next page.

A3

Q3. (a) (ii) Use the Convolution Theorem (see Table) with $F(s) = G(s) = 1/(s + 4)$ to show that the Inverse Laplace Transform of $1/(s + 4)^2$ is

$$f(t) = te^{-4t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) = e^{-4t}$$

By the convolution theorem

$$\mathcal{L}^{-1}\left(\frac{1}{(s+4)^2}\right) = \int_0^t e^{-4\tau} e^{-4(t-\tau)}\, d\tau$$

$$= e^{-4t} \int_0^t d\tau$$

$$= t e^{-4t}$$

Question 3 continued on next page.

A3

Q3. (b) Use the method of Laplace Transforms to solve the equation

$$y''(t) + 3y'(t) - 4y(t) = -5e^{-4t}$$

with Initial Conditions $y(0) = 1$, $y'(0) = -8$.

Let $\mathcal{L}(y(t)) = Y(s)$

Then $\mathcal{L}(y'' + 3y' - 4y)$

$$= [s^2 Y(s) - s y(0) - y'(0)]$$
$$+ 3[s Y(s) - y(0)] - 4Y(s)$$

So

$$(s^2 + 3s - 4)Y - s + 5 = -5\frac{1}{s+4}$$
$$(s+4)(s-1)Y = s - 5 - \frac{5}{s+4}$$
$$= \frac{s^2 - s - 25}{s+4}$$
$$Y = \frac{s^2 - s - 25}{(s-1)(s+4)^2}$$
$$= \frac{A}{s-1} + \frac{B}{s+4} + \frac{C}{(s+4)^2}$$

$\Rightarrow \quad s^2 - s - 25 = A(s+4)^2 + B(s-1)(s+4) + C(s-1)$

$\underline{s=1}$: $\quad -25 = 25A \Rightarrow A = -1$

$\underline{s=-4}$: $\quad -5 = -5C \Rightarrow C = 1$

Coeff. of s^2: $\quad 1 = A + B \Rightarrow B = 2$

So $Y(s) = \frac{-1}{s-1} + \frac{2}{s+4} + \frac{1}{(s+4)^2}$

and (using 3(a)(ii))

Table of Laplace Transforms see next page. TURN OVER

$$\underline{y(s) = -e^{t} + 2e^{-4t} + te^{-4t}}$$

A3

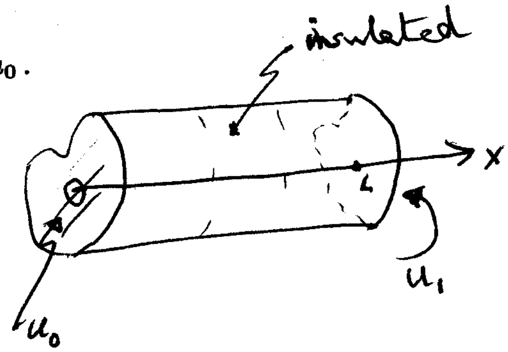
Q A3. (b) **Working space only.**

Question B1 see next page.

PART B

Q B1. (a) For times $t > 0$, a cylinder of iron of length L, with its curved sides insulated, lies along the X-axis with its planar face at $x = 0$ maintained at constant temperature u_0 and its planar face at $x = L$ maintained at constant temperature u_1. At time $t = 0$, the temperature distribution in the cylinder is a function $f(x)$, for $0 < x < L$. Write down the PDE, BCs and IC satisfied by the temperature $u(x,t)$ for $0 < x < L$ and $t > 0$, and deduce that the steady-state temperature $u^{SS}(x)$ which will be approached as $t \to \infty$ is

$$u^{SS}(x) = \left(\frac{u_1 - u_0}{L}\right) x + u_0 .$$



PDE: $u_t(x,t) = c^2 u_{xx}(x,t)$
$0 < x < L, \quad t > 0$

BCs: $u(0,t) = u_0 \quad t > 0$
$u(L,t) = u_1 \quad t > 0$

IC: $u(x,0) = f(x), \quad 0 < x < L.$

As $t \to \infty$, $u(x,t) \to u^{SS}(x)$
$\cancel{u^{SS}_t(x)} = c^2 u^{SS}_{xx}(x)$
$\quad 0$

$\Rightarrow \quad u^{SS}(x) = Ax + B$

BCs: $u^{SS}(0) = u_0 \Rightarrow \quad u_0 = 0 + B \Rightarrow u^{SS}(x) = Ax + u_0$
$u^{SS}(L) = u_1 \Rightarrow AL + u_0 = u_1 \Rightarrow A = \frac{u_1 - u_0}{L}$

So $\quad u^{SS}(x) = \left(\frac{u_1 - u_0}{L}\right) x + u_0$

Question B1 continued on next page.

Q B1. (b) Write $\hat{u}(x,t) = u(x,t) - u^{SS}(x)$, and find the PDE, BCs and IC satisfied by $\hat{u}(x,t)$. Work carefully through Fourier's Method of Separation of Variables and Superposition to obtain the solution $\hat{u}(x,t)$ of these equations in the case $f(x) = u_2$ (const.), and hence deduce that in this case

$$u(x,t) = \left(\frac{u_1 - u_0}{L}\right) x + u_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[u_2 - u_0 + (u_1 - u_2)(-1)^n]}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi c/L)^2 t}.$$

$\hat{u}(x,t) = u(x,t) - u^{SS}(x)$

Then PDE: $\hat{u}_t - c^2 \hat{u}_{xx} = (\cancel{u_t - c^2 u_{xx}}) - (\cancel{u^{SS}_t - c^2 u^{SS}_{xx}})$ ①
$= 0 \qquad 0 < x < L, \quad t > 0$

BCs ② $\hat{u}(0,t) = u(0,t) - u^{SS}(0) = u_0 - u_0 = 0, \quad t > 0.$

③ — $\hat{u}(L,t) = u(L,t) - u^{SS}(L) = u_1 - u_1 = 0, \quad t > 0$

IC ④ $\hat{u}(x,0) = u(x,0) - u^{SS}(x) = f(x) - u^{SS}(x) = \hat{f}(x)$, say.

Fourier's Method: Look for solutions of homogeneous equations ①, ②, ③ in separated form:

$\hat{u}(x,t) = F(x)\,G(t)$

② $\Rightarrow \quad 0 = F(0)G(t) \Rightarrow \quad G(t) = 0 \quad (\Rightarrow \hat{u}(x,t) = 0$, trivial$)$
or $\boxed{F(0) = 0}$ ⑤

③ $\Rightarrow \quad 0 = F(L)G(t) \Rightarrow \quad G(t) = 0 \quad (\Rightarrow \hat{u}(x,t) = 0$, trivial$)$
or $\boxed{F(L) = 0}$ ⑥

Question B1 continued on next page

Q B1. (b) **Working space only.**

Next, ① $\Rightarrow$ $F(x)G'(t) = c^2 F''(x)G(t)$

$\div\; c^2 F(x)G(t) \Rightarrow \dfrac{G'(t)}{c^2 G(t)} = \dfrac{F''(x)}{F(x)} = k$ (necessarily, const.)

$\Rightarrow \boxed{F''(x) = kF(x)}$ ⑦ and $\boxed{G'(t) = kc^2 G(t)}$ ⑧

Three cases to consider:

Case a) $k=0$ ⑦ $\Rightarrow F''(x) = 0 \Rightarrow F(x) = Ax + B$

Then ⑤ $\Rightarrow 0 = 0 + B \Rightarrow F(x) = Ax$

⑥ $\Rightarrow 0 = AL \Rightarrow A = 0 \Rightarrow F(x) = 0 \Rightarrow \hat{u}(x,t) = 0$ trivial

Case b) $k>0$ ($k = \mu^2$, say, $\mu > 0$)

⑦ $\Rightarrow F''(x) = \mu^2 F(x) \Rightarrow F(x) = A\cosh(\mu x) + B\sinh(\mu x)$

Then ⑤ $\Rightarrow 0 = A + 0 \Rightarrow F(x) = B\sinh(\mu x)$

⑥ $\Rightarrow 0 = B\sinh(\mu L) \Rightarrow B = 0 \Rightarrow F(x) = 0 \Rightarrow \hat{u}(x,t) =$ trivial

Case c) $k<0$ ($k = -p^2$, say, $p>0$)

⑦ $\Rightarrow F''(x) = -p^2 F(x) \Rightarrow F(x) = A\cos(px) + B\sin(px)$

Then ⑤ $\Rightarrow 0 = A + 0 \Rightarrow F(x) = B\sin(px)$

⑥ $\Rightarrow 0 = B\sin(pL) \Rightarrow B = 0$ ($\Rightarrow F(x) = 0 \Rightarrow \hat{u}(x,t) = 0$, trivial)

or $pL = n\pi \Rightarrow p = \dfrac{n\pi}{L}$ $n = 1, 2, 3, \ldots$

and $F(x) = B\sin\left(\dfrac{n\pi x}{L}\right)$. $\Downarrow$ $k = -\left(\dfrac{n\pi}{L}\right)^2$

Q B1. (b) **Working space only.**

For this value of k, ⑧ $\Rightarrow$ $G'(t) = -\left(\frac{n\pi c}{L}\right)^2 G(t)$

$\Rightarrow$ $G(t) = De^{-\left(\frac{n\pi c}{L}\right)^2 t}$

Combining: $\hat{u}(x,t) = P.D \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

or $\hat{u}_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

Following Fourier, assume gen. solution of ①, ②, ③ is

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

Then ④ $\Rightarrow$ $\hat{f}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ half-range sine series for $\hat{f}$.

$$\Rightarrow \quad B_n = \frac{2}{L}\int_0^L \hat{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

With $f(x) = u_2$, we have $\overline{\hat{f}(x) = u_2 - u^{ss}(x)}$

$$= (u_2 - u_0) + \left(\frac{u_0 - u_1}{L}\right) x$$

Then $B_n = \frac{2}{L}\left\{ \int_0^L (u_2 - u_0) \sin\left(\frac{n\pi x}{L}\right) dx + \left(\frac{u_0 - u_1}{L}\right)\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \right.$

$$= \frac{2}{L}\left\{ (u_2 - u_0)\left(-\frac{L}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{L}\right)\right]_0^L \right.$$

$$+ \left(\frac{u_0 - u_1}{L}\right)\left[-x\left(\frac{L}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \right]$$

TURN OVER

Q B1. (b) **Working space only.**

$$B_n = \frac{2}{L}\left\{ (u_2 - u_0)\left(-\frac{L}{n\pi}\right)\left[(-1)^n - 1\right] + \frac{(u_0 - u_1)}{L}(-L)\left(\frac{L}{n\pi}\right)(-1)^n + \left(\frac{L}{n\pi}\right)^2 \left[\cancel{\sin\left(\frac{n\pi x}{L}\right)}\right]_0^L \right\}$$

$$= \frac{2}{n\pi}(u_2 - u_0) - \frac{2}{n\pi}\left[(u_2 - u_0) + (u_0 - u_1)\right](-1)^n$$

$$= \frac{2}{n\pi}\left[(u_2 - u_0) + (u_1 - u_2)(-1)^n\right]$$

Then
$$\hat{u}(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left[(u_2 - u_0) + (u_1 - u_2)(-1)^n\right]}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

and
$$u(x,t) = \hat{u}(x,t) + u^{ss}(x)$$

$$= \left(\frac{u_1 - u_0}{L}\right)x + u_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left[u_2 - u_0 + (u_1 - u_2)(-1)^n\right]}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

Question B2 see next page.

Q B2. (a) Use the formulas

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)]; \quad \sin(A)\cos(B) = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

to deduce that if $n \neq 1$, then

$$\int \sin(nx)\sin(x)\,dx = \frac{1}{2}\left(\frac{1}{n-1}\sin[(n-1)x] - \frac{1}{n+1}\sin[(n+1)x]\right)$$
$$\int \cos(nx)\sin(x)\,dx = \frac{1}{2}\left(\frac{1}{n-1}\cos[(n-1)x] - \frac{1}{n+1}\cos[(n+1)x]\right),$$

and that

$$\int \sin^2(x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin[2x]\right); \quad \int \cos(x)\sin(x)\,dx = -\frac{1}{4}\cos(2x).$$

$$\int \sin(nx)\sin(x)\,dx = \frac{1}{2}\int\left[\cos[(n-1)x] - \cos[(n+1)x]\right]dx$$

$$n \neq 1 \Rightarrow \quad \frac{1}{2}\left(\frac{1}{n-1}\sin[(n-1)x] - \frac{1}{n+1}\sin[(n+1)x]\right)$$

$$n = 1 \Rightarrow \quad \frac{1}{2}\int[1 - \cos(2x)]dx = \frac{1}{2}\left[x - \frac{1}{2}\sin[2x]\right]$$

$$\int \cos(nx)\sin(x)\,dx = \frac{1}{2}\int\left[\sin[(n+1)x] + \sin[(1-n)x]\right]dx.$$

$$n \neq 1 \Rightarrow \quad -\frac{1}{2}\left[\frac{1}{n+1}\cos[(n+1)x] + \frac{1}{1-n}\cos[(1-n)x]\right]$$

$$= +\frac{1}{2}\left[\frac{1}{n-1}\cos[(n-1)x] - \frac{1}{n+1}\cos[(n+1)x]\right]$$

$$n = 1 \Rightarrow \quad \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x).$$

Question B2 continued on next page.

Q B2.

(b) Show that the Fourier Series corresponding to the function defined by

$$f(x) = 0, \qquad -\pi < x < 0; \qquad f(x) = \sin(x), \qquad 0 < x < \pi;$$
$$\text{and} \qquad f(x + 2\pi) = f(x), \qquad -\infty < x < \infty,$$

is

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\left(\frac{1}{(1)(3)}\cos(2x) + \frac{1}{(3)(5)}\cos(4x) + \cdots\right).$$

Fourier Series is

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Question B2 continued on next page. TURN OVER

Q B2. (b) **Working space only**

In this case:

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)\,dx = -\frac{1}{2\pi}\left[\cos(x)\right]_0^{\pi} = -\frac{1}{2\pi}\left[-1-1\right] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$n \neq 1 \Rightarrow \quad a_n = \frac{1}{2\pi}\left\{\left[\frac{1}{(n-1)}\cos[(n-1)x] - \frac{1}{(n+1)}\cos[(n+1)x]\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n-1}\left[\cos[(n-1)\pi]-1\right] - \frac{1}{n+1}\left[\cos[(n+1)\pi]-1\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n-1}\left[(-1)^{n-1}-1\right] - \frac{1}{n+1}\left[(-1)^{n+1}-1\right]\right\}$$

$$= \frac{1}{2\pi}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\left[(-1)^{n+1}-1\right]$$

$$= \frac{1}{2\pi}\,\frac{(n+1)-(n-1)}{n^2-1}\left[(-1)^{n+1}-1\right]$$

$$= \frac{1}{\pi}\,\frac{\left[(-1)^{n+1}-1\right]}{(n-1)(n+1)}$$

$$n = 1 \Rightarrow \quad a_1 = \frac{1}{\pi}\cdot -\frac{1}{4}\left[\cos(2x)\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx$$

$$n \neq 1 \Rightarrow \quad b_n = \frac{1}{2\pi}\left\{\left[\frac{1}{(n-1)}\sin[(n-1)x] - \frac{1}{n+1}\sin[(n+1)x]\right]_0^{\pi}\right\}$$

$$= 0$$

Question B2 continued on next page.

Q B2. (b) **Working space only**

$n=1 \Rightarrow b_1 = \frac{1}{\pi} \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi = \frac{1}{2}.$

So series is

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{\left[(-1)^{n+1} - 1\right]}{(n-1)(n+1)} \cos(nx)$$

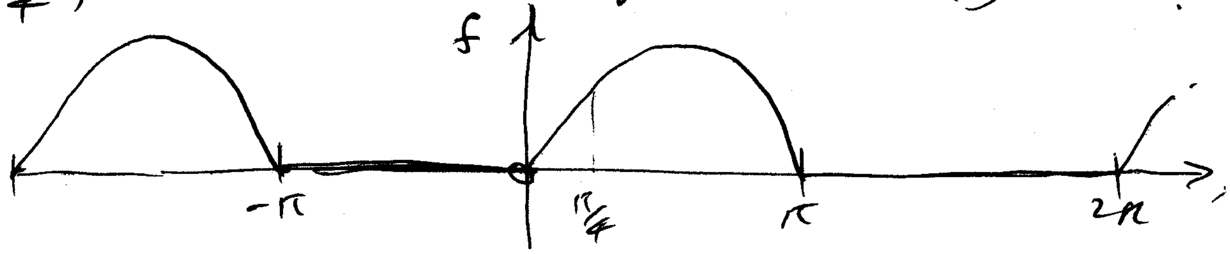
$$= \frac{1}{\pi} + \frac{1}{2}\sin(x) + \frac{1}{\pi}\left\{\frac{-2}{(1)(3)}\cos(2x) - \frac{2}{(3)(5)}\cos(4x) - \cdots\right\}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\left(\frac{1}{(1)(3)}\cos(2x) + \frac{1}{(3)(5)}\cos(4x) + \cdots\right)$$

Question B2 continued on next page.

Q B2. (c) By considering the value to which the series should converge at $x = \pi/4$, deduce that
$$\frac{1}{(3)(5)} - \frac{1}{(7)(9)} + \frac{1}{(11)(13)} \cdots = \frac{\pi}{4\sqrt{2}} - \frac{1}{2}.$$

At $x = \frac{\pi}{4}$, series should converge to $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.



So
$$\frac{1}{\sqrt{2}} = \frac{1}{\pi} + \frac{1}{2}\sin\left(\frac{\pi}{4}\right) - \frac{2}{\pi}\left(\frac{1}{(1)(3)}\cos\left(\frac{\pi}{2}\right) + \frac{1}{(3)(5)}\cos(\pi) + \frac{1}{(5)(7)}\cos\left(\frac{3\pi}{2}\right) + \frac{1}{(7)(9)}\cos(2\pi) + \cdots\right)$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\pi} + \frac{1}{2\sqrt{2}} - \frac{2}{\pi}\left[0 - \frac{1}{(3)(5)} + 0 + \frac{1}{(7)(9)} + 0 - \cdots\right]$$

$$\left(\frac{1}{2\sqrt{2}} - \frac{1}{\pi}\right) = \frac{2}{\pi}\left[\frac{1}{(3)(5)} - \frac{1}{(7)(9)} + \frac{1}{(11)(13)} - \cdots\right]$$

$$\Rightarrow \frac{1}{(3)(5)} - \frac{1}{(7)(9)} + \frac{1}{(11)(13)} - \cdots = \frac{\pi}{2}\left(\frac{1}{2\sqrt{2}} - \frac{1}{\pi}\right)$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{1}{2}$$

Question B3 see next page. TURN OVER

Q B3. (a) Show that the function $G(x-y,t)$ defined by

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-(x-y)^2/(4c^2t)}$$

satisfies the 1-dimensional Heat Equation:

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

$$G_t(x-y,t) = \frac{1}{\sqrt{4\pi c^2}} \Big\{ -\tfrac{1}{2} t^{-\frac{3}{2}} e^{-(x-y)^2/(4c^2t)} + t^{-\frac{1}{2}} \frac{(x-y)^2}{4c^2t^2} e^{-(x-y)^2/4c^2t}$$

$$= \frac{1}{\sqrt{4\pi c^2}} \Big\{ -\tfrac{1}{2} t^{-\frac{3}{2}} + \frac{(x-y)^2}{4c^2 t^{5/2}} \Big\} e^{-(x-y)^2/4c^2t}$$

$$G_x(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}} \Big\{ \frac{-2(x-y)}{4c^2 t} e^{-(x-y)^2/4c^2t} \Big\}$$

$$= -\frac{1}{2c^2} \frac{t^{-3/2}}{\sqrt{4\pi c^2}} \Big\{ (x-y)\, e^{-(x-y)^2/4c^2t} \Big\}$$

$$G_{xx}(x-y,t) = -\frac{1}{2c^2} \frac{t^{-3/2}}{\sqrt{4\pi c^2}} \Big\{ 1 - \frac{2(x-y)^2}{4c^2 t} \Big\} e^{-(x-y)^2/4c^2t}$$

$$c^2 G_{xx}(x-y,t) = \frac{1}{\sqrt{4\pi c^2}} \Big\{ -\tfrac{1}{2} t^{-3/2} + \frac{(x-y)^2}{4c^2 t^{5/2}} \Big\} e^{-(x-y)^2/4c^2t}$$

$$= G_t(x-y,t)$$

Question B3 continued on next page.

2

Q B3. (a) **Working space only**

Question B3 continued on next page.

Q B3. (b) You are given (**no need to check**) that $u(x,t)$ defined by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

satisfies the 1-dimensional Heat Equation for $-\infty < x < \infty$ and $t > 0$, and also the initial condition

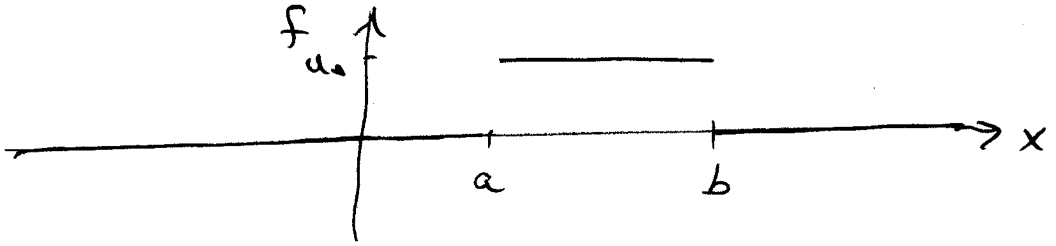
$$\lim_{t \to 0_+} u(x,t) = f(x), \qquad -\infty < x < \infty.$$

Show that in the case when $f(x) = u_0$ (const.) for $a < x < b$ and $f(x) = 0$ otherwise, this gives

$$u(x,t) = \frac{1}{2} u_0 \left(\operatorname{erf}\left(\frac{x-a}{\sqrt{4c^2 t}}\right) - \operatorname{erf}\left(\frac{x-b}{\sqrt{4c^2 t}}\right)\right),$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv.$$



$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_a^b u_0\, e^{-(x-y)^2/4c^2 t}\, dy$$

Put $v = \frac{x-y}{\sqrt{4c^2 t}} \leftrightarrow y = x - \sqrt{4c^2 t}\, v$

$dy = -\sqrt{4c^2 t}\, dv$

$y = a \leftrightarrow v = \frac{x-a}{\sqrt{4c^2 t}}$

$y = b \leftrightarrow v = \frac{x-b}{\sqrt{4c^2 t}}$

Question B3 continued on next page.

Q B3. (b) **Working space only**

Then
$$u(x,t) = \frac{u_0}{\sqrt{4\pi c^2 t}} \int_{\frac{x-a}{\sqrt{4c^2t}}}^{\frac{x-b}{\sqrt{4c^2t}}} e^{-v^2} \left(-\sqrt{4c^2t}\right) dv$$

$$= \frac{u_0}{\sqrt{\pi}} \int_{\frac{x-b}{\sqrt{4c^2t}}}^{\frac{x-a}{\sqrt{4c^2t}}} e^{-v^2} dv$$

$$= \tfrac{1}{2} u_0 \left\{ \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-a}{\sqrt{4c^2t}}} e^{-v^2} dv - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-b}{\sqrt{4c^2t}}} e^{-v^2} dv \right\}$$

$$= \tfrac{1}{2} u_0 \left[\operatorname{erf}\left(\frac{x-a}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{x-b}{\sqrt{4c^2t}}\right) \right]$$

Question B3 continued on next page.

Q B3.

(c) Check that there is no flux of heat in the x-direction at the point $x = (a+b)/2$.

Heat flux vector is $\underset{\sim}{J} = -K\, u_x \underset{\sim}{i}$

So: Need to show $u_x\left(\frac{a+b}{2}, t\right) = 0$.

By chain rule:

$$u_x(x,t) = \frac{1}{2}u_0 \left\{ \frac{\partial}{\partial x}\left(\frac{x-a}{\sqrt{4c^2t}}\right) \operatorname{erf}'\left(\frac{x-a}{\sqrt{4c^2t}}\right) - \frac{\partial}{\partial x}\left(\frac{x-b}{\sqrt{4c^2t}}\right) \operatorname{erf}'\left(\frac{x-b}{\sqrt{4c^2t}}\right) \right\}$$

$$= \frac{1}{2}u_0 \frac{1}{\sqrt{4c^2t}} \left\{ \operatorname{erf}'\left(\frac{x-a}{\sqrt{4c^2t}}\right) - \operatorname{erf}'\left(\frac{x-b}{\sqrt{4c^2t}}\right) \right\}$$

$\operatorname{erf}'(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}$, so

$$u_x(x,t) = \frac{u_0}{\sqrt{4\pi c^2 t}} \left\{ e^{-\frac{(x-a)^2}{4c^2t}} - e^{-\frac{(x-b)^2}{4c^2t}} \right\}$$

and

$$u_x\left(\frac{a+b}{2}, t\right) = \frac{u_0}{\sqrt{4\pi c^2 t}} \left\{ e^{-\left(\frac{b-a}{2}\right)^2/4c^2t} - e^{-\left(\frac{a-b}{2}\right)^2/4c^2t} \right\} = 0$$

Table of Laplace Transforms see next page.

TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}f(t)$	$F(s-a)$
$t\,f(t)$	$-\frac{d}{ds}F(s)$
$g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t > c. \end{cases} = f(t-c)u(t-c)$	$e^{-sc}F(s) \quad c > 0$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$f(t+p) = f(t), \quad t > 0$	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$