MATH 2010/2100 Solutions 1 — 2007/1 AJB

K4.1 #12* (a) $y'' + 2y' - 24y = 0$

Put $y_1 = y$, $y_2 = y'$

$$\Rightarrow \quad y_1' = y' = y_2, \; y_2' = y'' = 24y - 2y' = 24y_1 - 2y_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \circledast \text{ i.e. } \underset{\sim}{y}' = A\underset{\sim}{y}, \; A = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} \quad ①$$

E'value condition: ($\det(A - \lambda I) = 0$)

$$\begin{vmatrix} 0-\lambda & 1 \\ 24 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+2) - 24 = 0$$
$$\Rightarrow \lambda^2 + 2\lambda - 24 = 0$$
$$\Rightarrow (\lambda+6)(\lambda-4) = 0$$

E'values: $\underline{\lambda_1 = -6, \lambda_2 = 4}$ ①

Eigenvectors: $\lambda_1 = -6$

$$\begin{bmatrix} 0+6 & 1 \\ 24 & -2+6 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} 6u + v = 0 \\ 24u + 4v = 0 \end{array}\right\} \Rightarrow v = -6u$$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ ①

$\lambda_2 = 4$

$$\begin{bmatrix} 0-4 & 1 \\ 24 & -2-4 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} -4u + v = 0 \\ 24u - 6v = 0 \end{array}\right\} \Rightarrow v = 4u$$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ ①

We know ~~that~~ $\underset{\sim}{x}^{(1)} e^{\lambda_1 t}$ and $\underset{\sim}{x}^{(2)} e^{\lambda_2 t}$ are two solutions of $\circledast$. The general solution is then

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{-6t} + \beta \underset{\sim}{x}^{(2)} e^{4t}$$

i.e.

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -6 \end{pmatrix} e^{-6t} + \beta \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} \quad ①$$

[Note from first components that
$$y_1(t) \equiv y(t) = \alpha e^{-6t} + \beta e^{4t}$$]

(b) $y'' + 2y' - 24y = 0$ (**)

Try $y = e^{\lambda t}$ ($\Rightarrow y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$)

$\Rightarrow \lambda^2 \cancel{e^{\lambda t}} + 2\lambda \cancel{e^{\lambda t}} - 24 \cancel{e^{\lambda t}} = 0.$

$(\lambda^2 + 2\lambda - 24) = 0$ charac. quadratic

$\Rightarrow (\lambda + 6)(\lambda - 4) = 0$

Roots: $\lambda_1 = -6$, $\lambda_2 = 4$. (1)

We know then ~~that~~ e^{-6t} and e^{4t} are two solutions of (**). The general solution is then

$$y(t) = \alpha e^{-6t} + \beta e^{4t}$$ as before (1)

K4.1 #13 (a) $y'' - y' = 0$

Let $y_1 = y$, $y_2 = y'$

$\Rightarrow y_1' = y' = y_2$, $y_2' = y'' = y' = y_2$

So $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (*) i.e. $\underset{\sim}{y}' = A\underline{y}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

E'Value condition: ($\det(A - \lambda I) = 0$)

$\begin{vmatrix} 0-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 1) = 0$

$\Rightarrow$ E'Values $\lambda_1 = 0$, $\lambda_2 = 1$.

E'Vectors: $\lambda_1 = 0$

$\begin{bmatrix} 0-0 & 1 \\ 0 & 1-0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v = 0 \\ v = 0 \end{cases} \Rightarrow v = 0$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1$

$\begin{bmatrix} 0-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -u + v = 0 \\ 0 = 0 \end{cases} \Rightarrow u = v$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We know then ~~that~~ $\underset{\sim}{x}^{(1)} e^{\lambda_1 t}$ and $\underset{\sim}{x}^{(2)} e^{\lambda_2 t}$ are two solutions of (*). Then the general solution is

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + \beta \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$$

i.e. $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \equiv \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t.$

(b) $y'' - y' = 0$ (**)

Try $y = e^{\lambda t}$ $(\Rightarrow y' = \lambda e^{\lambda t},\ y'' = \lambda^2 e^{\lambda t})$

$\Rightarrow \lambda^2 \cancel{e^{\lambda t}} - \lambda \cancel{e^{\lambda t}} = 0 \Rightarrow \lambda^2 - \lambda = 0$

$\Rightarrow \lambda(\lambda - 1) = 0$ charac. quadratic.

Roots: $\lambda_1 = 0,\ \lambda_2 = 1$

We know then ~~that~~ $e^{0t} = 1$ and e^t are two solutions of (**). The general solution is then

$$y(t) = \alpha + \beta e^t, \quad \text{as before}$$

K4.5 #14 (a) $y'' + 15y' + 50y = 0$

Let $y_1 = y,\ y_2 = y'$

$\Rightarrow y_1' = y' = y_2,\ y_2' = y'' = -50y - 15y' = -50y_1 - 15y_2$

So $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (*) i.e. $\underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix}$

E'Value condition:
($\det(A - \lambda I) = 0$)

$\begin{vmatrix} 0-\lambda & 1 \\ -50 & -15-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda + 15) + 50 = 0$

$\Rightarrow \lambda^2 + 15\lambda + 50 = 0$

$\Rightarrow (\lambda + 5)(\lambda + 10) = 0$

$\Rightarrow$ E'Values $\lambda_1 = -5,\ \lambda_2 = -10.$

E'Vectors: $\lambda_1 = -5$

$\begin{bmatrix} 0+5 & 1 \\ -50 & -15+5 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 5u + v = 0 \\ -50u - 10v = 0 \end{matrix}\right\} \Rightarrow v = -5u$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

$\lambda_2 = -10$

$\begin{bmatrix} 0+10 & 1 \\ -50 & -15+10 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 10u + v = 0 \\ -50u - 5v = 0 \end{matrix}\right\} \Rightarrow v = -10u$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$

Now we know ~~that~~ $\underset{\sim}{x}^{(1)} e^{\lambda_1 t}$ and $\underset{\sim}{x}^{(2)} e^{\lambda_2 t}$ are two solutions of (*). The general solution is then

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + \beta \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$$

i.e.

$$\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t} + \beta \begin{pmatrix} 1 \\ -10 \end{pmatrix} e^{-10t}.$$

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \Rightarrow y(t) = \alpha e^{-5t} + \beta e^{-10t}.$$

(b) $y'' + 15y' + 50y = 0$ (**)

Try $y = e^{\lambda t}$ $(\Rightarrow y' = \lambda e^{\lambda t},\ y'' = \lambda^2 e^{\lambda t})$

$\Rightarrow \lambda^2 \cancel{e^{\lambda t}} + 15\lambda \cancel{e^{\lambda t}} + 50\cancel{e^{\lambda t}} = 0$

$\Rightarrow \lambda^2 + 15\lambda + 50 = 0$ charac. quadratic.

$\Rightarrow (\lambda+5)(\lambda+10) = 0$

$\Rightarrow$ Roots $\lambda_1 = -5,\ \lambda_2 = -10$.

We know ~~then that~~ e^{-5t} and e^{-10t} are two solutions of (**). The general solution is then

$$y(t) = \alpha e^{-5t} + \beta e^{-10t}$$ as before

K4.3, #4

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} 9 & \frac{27}{2} \\ \frac{3}{2} & 9 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \ (*) \quad \text{ie } \underset{\sim}{y}' = A\underset{\sim}{y},\quad A = \begin{bmatrix} 9 & \frac{27}{2} \\ \frac{3}{2} & 9 \end{bmatrix}$$

E'Value condition: $(\det(A - \lambda I) = 0)$

$$\begin{vmatrix} 9-\lambda & \frac{27}{2} \\ \frac{3}{2} & 9-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-9)(\lambda-9) - \frac{81}{4} = 0$$

$$\Rightarrow \lambda^2 - 18\lambda + \frac{243}{4} = 0$$

$$\Rightarrow (\lambda - \tfrac{9}{2})(\lambda - \tfrac{27}{2}) = 0$$

$$\Rightarrow \text{E'values } \lambda_1 = \tfrac{9}{2},\ \lambda_2 = \tfrac{27}{2}$$

E'Vectors: $\lambda_1 = \frac{9}{2}$

$$\begin{bmatrix} 9-\frac{9}{2} & \frac{27}{2} \\ \frac{3}{2} & 9-\frac{9}{2} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} \frac{9}{2} & \frac{27}{2} \\ \frac{3}{2} & \frac{9}{2} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} 9u + 27v = 0 \\ 3u + 9v = 0 \end{matrix}\right\} \Rightarrow u = -3v$$

Choose $v = 1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{27}{2}$

$$\begin{bmatrix} 9-\frac{27}{2} & \frac{27}{2} \\ \frac{3}{2} & 9-\frac{27}{2} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} -\frac{9}{2} & \frac{27}{2} \\ \frac{3}{2} & -\frac{9}{2} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} -9u + 27v = 0 \\ 3u - 9v = 0 \end{matrix}\right\} \Rightarrow u = 3v$$

Choose $v = 1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Now we know that $\underset{\sim}{x}^{(1)} e^{\lambda_1 t}$ and $\underset{\sim}{x}^{(2)} e^{\lambda_2 t}$ are two solutions of (*)
The general solution is then

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + \beta \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$$

ie. $$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{9t/2} + \beta \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{27t/2}$$

(1.5)

K4.3 #10* $\underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, $\underset{\sim}{y}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

E'value condition: ($\det(A - \lambda I) = 0$)

$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 - 4 = 0$
$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$
$\Rightarrow (\lambda-3)(\lambda+1) = 0$
$\Rightarrow$ E'Values $\lambda_1 = 3$, $\lambda_2 = -1$. (1)

E'Vectors: $\lambda_1 = 3$

$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -2u+v=0 \\ 4u-2v=0 \end{matrix}\right\}$

$\Rightarrow v = 2u$

Choose $v=1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (1)

$\lambda_2 = -1$ $\begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 2u+v=0 \\ 4u+2v=0 \end{matrix}\right\}$

$\Rightarrow v = -2u$

choose $u=1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (1)

General solution is then

$\underset{\sim}{y}(t) = \alpha\, \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + \beta\, \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$
$= \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ (1)

IC: $\underset{\sim}{y}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} 1 + \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix} 1$

$\left.\begin{matrix} 1 = \alpha + \beta \\ 6 = 2\alpha - 2\beta \end{matrix}\right\} \Rightarrow \alpha = 2, \ \beta = -1.$

So $\underset{\sim}{y}(t) = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

i.e. $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ (2)

K 4.3 #12* $\underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $\underset{\sim}{y}(0) = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

E'value condition: ($\det(A - \lambda I) = 0$)

$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-3)^2 - 4 = 0$
$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$
$\Rightarrow (\lambda-5)(\lambda-1) = 0$

$\Rightarrow$ E'values: $\lambda_1 = 5$, $\lambda_2 = 1$. (1)

E'Vectors: $\lambda_1 = 5$. $\begin{bmatrix} 3-5 & 2 \\ 2 & 3-5 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{r} -2u + 2v = 0 \\ 2u - 2v = 0 \end{array}\right\} \Rightarrow u = v.$$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (1)

$\lambda_2 = 1$ $\begin{bmatrix} 3-1 & 2 \\ 2 & 3-1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{r} 2u + 2v = 0 \\ 2u + 2v = 0 \end{array}\right\} \Rightarrow u = -v.$$

Choose $u = 1$, then e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (1)

General solution is then

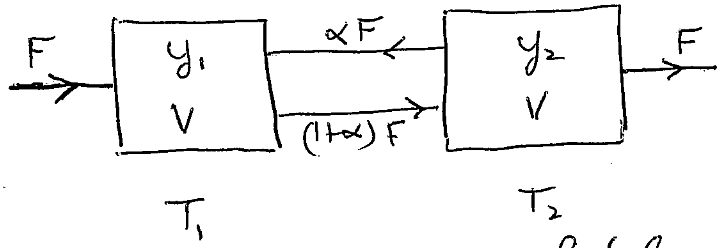
$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + \beta \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$$
$$= \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}. \quad (1)$$

IC: $\underset{\sim}{y}(0) = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 7 \\ 7 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{r} 7 = \alpha + \beta \\ 7 = \alpha - \beta \end{array}\right\} \Rightarrow \alpha = 7, \ \beta = 0.$$

So solution is $\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} e^{5t}$ (2)

K 4.3 #18



Let $y_1(t)$, $y_2(t)$ be masses of fertilizer in T_1, T_2 at time t.

Let $c_1(t)$, $c_2(t)$ be concentrations of solutions in T_1, T_2 at time t.

Let V be volume of T_1, T_2.

Then $c_1(t) = \dfrac{y_1(t)}{V}$, $c_2(t) = \dfrac{y_2(t)}{V}$.

(tanks are well-stirred!)

Let flow rates (volume/unit time) be as shown ($\alpha > 0$, α const.)

In time-interval t to $t+\delta t$ (1.7)

$$\delta y_1 = \underbrace{\alpha F c_2(t)\,\delta t}_{\text{mass flow into from 2}} - \underbrace{(1+\alpha) F c_1(t)\,\delta t}_{\text{mass flow out of 1 into 2}}$$

$$= \frac{\alpha F y_2(t)}{V}\delta t - \frac{(1+\alpha)F}{V} y_1(t)\,\delta t$$

Similarly

$$\delta y_2 = \underbrace{(1+\alpha) F c_1(t)\,\delta t}_{\text{mass flow into 2 from 1.}} - \underbrace{\alpha F c_2(t)\,\delta t}_{\text{mass flow out of 2 into 1}} - \underbrace{F c_2(t)\,\delta t}_{\text{mass flow out of 2 to Right}}$$

$$= \frac{(1+\alpha)F}{V} y_1(t)\,\delta t - \frac{(1+\alpha)F}{V} y_2(t)\,\delta t$$

Dividing by δt and letting $\delta t \to 0$ we get

$$\frac{dy_1}{dt} = \frac{\alpha F}{V} y_2 - \frac{(1+\alpha)F}{V} y_1$$

$$\frac{dy_2}{dt} = \frac{(1+\alpha)F}{V} y_1 - \frac{(1+\alpha)F}{V} y_2$$

[Note dimensions: $[\alpha]=1$, $[F]=L^3T^{-1}$, $[V]=L^3$, $[y_1]=M$
$[y_2]=M$, $[y_1']=[y_2']=MT^{-1}$ — all matches.
Note the desirability of not putting in numbers until the very end – otherwise can't see dimensions.]

Now
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} -\frac{(1+\alpha)F}{V} & \frac{\alpha F}{V} \\ \frac{(1+\alpha)F}{V} & -\frac{(1+\alpha)F}{V} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{bmatrix} -0.16 & 0.04 \\ 0.16 & -0.16 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4\beta & \beta \\ 4\beta & -4\beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $\beta = 0.04$ (minutes^{-1})

down to here

$$\underset{\sim}{y}' = A\underset{\sim}{y}, \quad A = \begin{bmatrix} -4\beta & \beta \\ 4\beta & -4\beta \end{bmatrix}$$

E'value condition:
(det $(A-\lambda I) = 0$)

$$\begin{vmatrix} -4\beta-\lambda & \beta \\ 4\beta & -4\beta-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+4\beta)^2 - 4\beta^2 = 0$$

$$\Rightarrow \lambda^2 + 8\beta\lambda + 12\beta^2 = 0 \Rightarrow (\lambda+6\beta)(\lambda+2\beta) = 0$$

$\Rightarrow$ E'values $\lambda_1 = -2\beta, \quad \lambda_2 = -6\beta$

E'vectors: $\lambda_1 = -2\beta$

$$\begin{bmatrix} -4\beta+2\beta & \beta \\ 4\beta & -4\beta+2\beta \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \beta\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2u + v &= 0 \\ 4u - 2v &= 0 \end{aligned}\right\} \Rightarrow v = 2u$$

Choose $u = 1$, then $v = 2$ and e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -6\beta$

$$\begin{bmatrix} -4\beta+6\beta & \beta \\ 4\beta & -4\beta+6\beta \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \beta\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2u + v &= 0 \\ 4u + 2v &= 0 \end{aligned}\right\} \Rightarrow v = -2u$$

Choose $u = 1$, then $v = -2$ and e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

General solution is then $\underset{\sim}{y}(t) = c_1 \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + c_2 \underset{\sim}{x}^{(2)} e^{\lambda_2 t}$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2\beta t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6\beta t}$$

IC: $\underset{\sim}{y}(0) = \begin{pmatrix} 100 \\ 40 \end{pmatrix}$, so $\begin{pmatrix} 100 \\ 40 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 100 \\ 2c_1 - 2c_2 &= 40 \end{aligned}\right\} \Rightarrow c_1 = 60, \; c_2 = 40.$$

So $\underset{\sim}{y}(t) = 60\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2\beta t} + 40\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6\beta t}$

$$\Rightarrow \left.\begin{aligned} y_1(t) &= 60\, e^{-0.08t} + 40\, e^{-0.24t} \\ y_2(t) &= 120\, e^{-0.08t} - 80\, e^{-0.24t} \end{aligned}\right\} \text{(1b)}.$$

[Note ~~that~~ (βt) is dimensionless: $[\beta t] = 1$, as must be since we can ~~form~~ $e^{\beta t} = 1 + \beta t + \frac{1}{2}(\beta t)^2 + \ldots$. We could not add these terms together if (βt) were not dimensionless!]