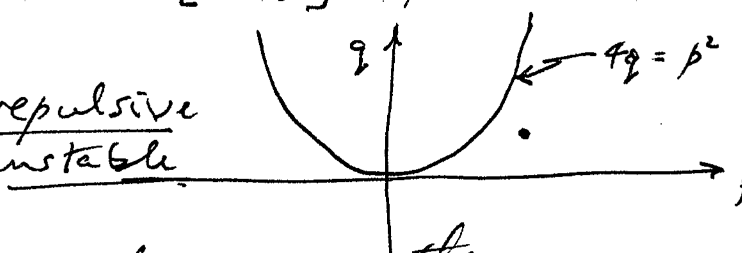
MATH 2010/2100 Solutions 2 AJJB

K4.4 #2* $\quad \underset{\sim}{y}' = A\underset{\sim}{y}, \quad A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow p = 7, \ q = 12, \ \Delta = 49 - 48 = 1$



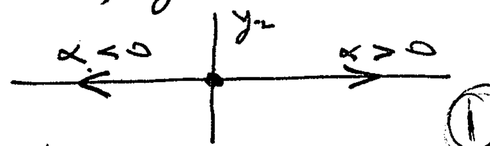
Improper, repulsive node — unstable 2

E'values and e'vectors are obvious in this case, as A is diagonal. $\lambda_1 = 4, \ \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 3, \ x^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 1

[If you can't see it immediately, go through the usual calculation of $\det(A - \lambda I) = 0$ etc.]

General solution: $\underset{\sim}{y}(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$ 1

Special trajectories: $\beta = 0 \Rightarrow y_2(t) = 0, \ y_1(t) = \alpha e^{4t}$



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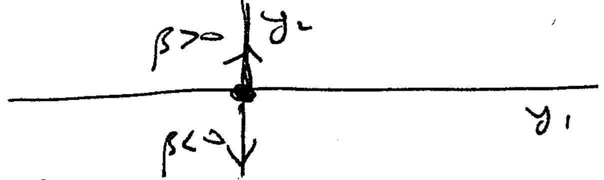
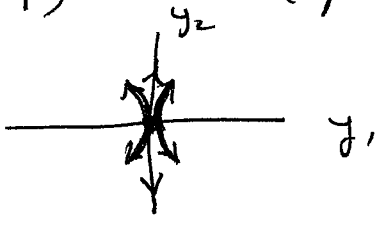
$\alpha = 0 \Rightarrow y_1(t) = 0, \ y_2(t) = \beta e^{3t}$



As $t \to -\infty$, every $\underset{\sim}{y}(t) \approx \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$ (provided $\beta \neq 0$)

So:— near origin, trajectories are tangent to $y_1 = 0$.



Also have special trajectory $\alpha = \beta = 0$ i.e. $\underset{\sim}{y} = \underset{\sim}{0}$

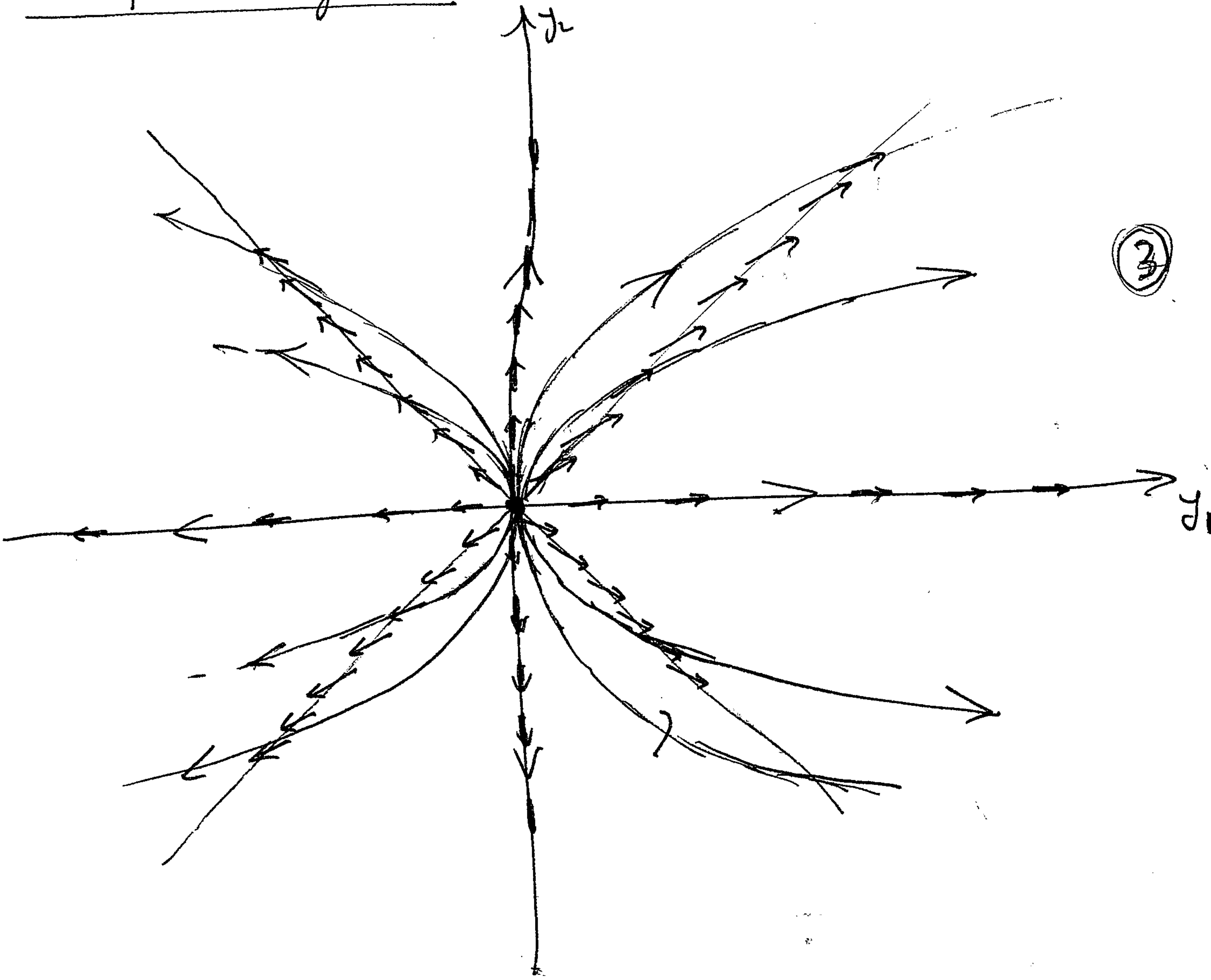
Next, slope equation: $\dfrac{dy_2}{dy_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{3y_2}{4y_1}$

$$= \begin{cases} 0 & \text{on } y_2 = 0 \\ \pm\infty & \text{on } y_1 = 0 \\ 3/4 & \text{on } y_2 = y_1 \\ -3/4 & \text{on } y_2 = -y_1. \end{cases}$$ 2

Also $\frac{dy_2}{dt} = 3y_2$. So on $y_1 = 0$ or $y_2 = y_1$ or $y_2 = -y_1$, y_2 is increasing where $y_2 > 0$, and decreasing where $y_2 < 0$.

$\frac{dy_1}{dt} = 4y_1$. So on $y_2 = 0$ or $y_2 = y_1$ or $y_2 = -y_1$, y_1 is increasing where $y_1 > 0$, and decreasing where $y_1 < 0$.

Putting all together:-



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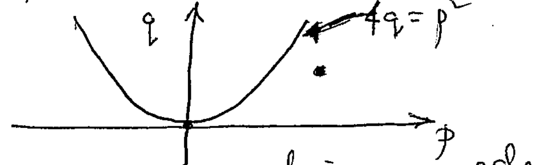
MATH 2010 Solutions 2

AJB

K 4.4 #3: $\underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow p = 4,\ q = 3,\ \Delta = 16 - 12 = 4$



Improper, repulsive node
unstable.

E' Values: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-2)^2 - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow (\lambda - 3)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = 3,\ \lambda_2 = 1.$

E' Vectors: $\underline{\lambda_1 = 3.}$ $\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -u+v=0 \\ u-v=0 \end{matrix}\right\} \Rightarrow u = v$

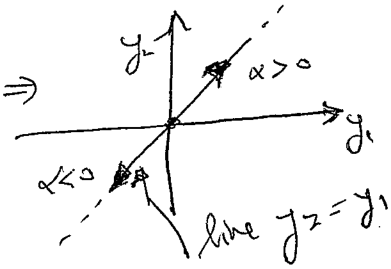
choosing $u = 1 \Rightarrow v = 1 \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 1}$ $\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} u+v=0 \\ u+v=0 \end{matrix}\right\} \Rightarrow v = -u$

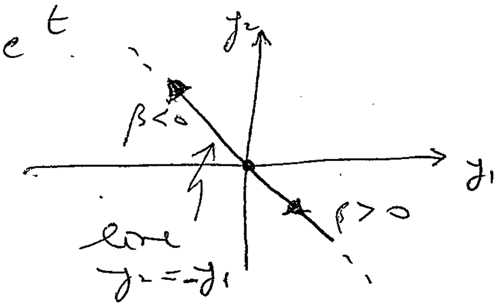
choosing $u = 1 \Rightarrow v = -1 \Rightarrow \underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution: $\underset{\sim}{y}(t) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}$

Special trajectories: $\beta = 0$: $y_1(t) = y_2(t) = \alpha e^{3t} \Rightarrow$



$\alpha = 0$: $y_2(t) = -y_1(t) = -\beta e^{t}$



Note directions!

[Also $\alpha = \beta = 0$: Origin $\underline{y} = \underline{0}$ is itself a trajectory)

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{y_1+2y_2}{2y_1+y_2}$$

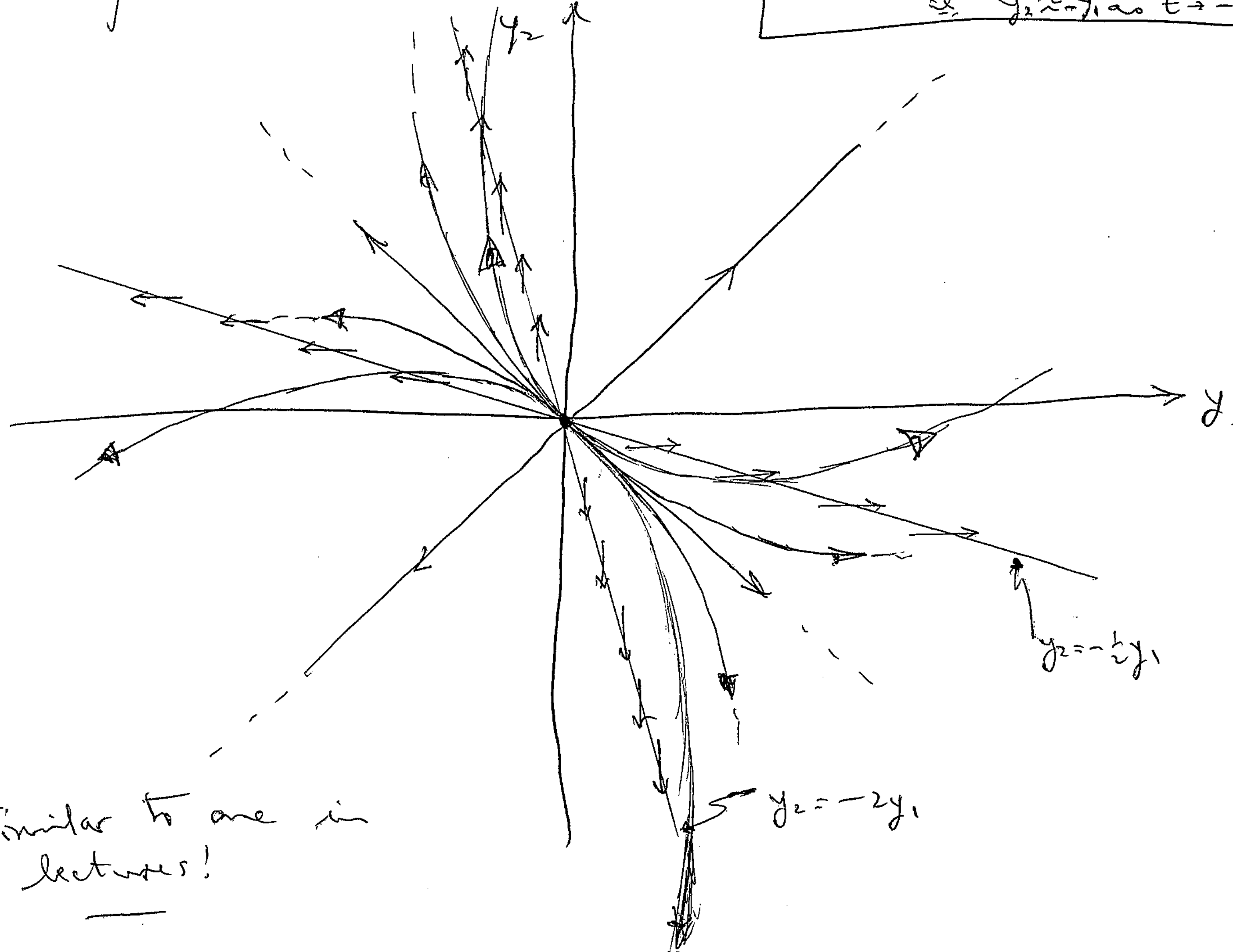
$$= \begin{cases} 0 & \text{on } y_2 = -\tfrac{1}{2}y_1 \\ \pm\infty & \text{on } y_2 = -2y_1 \\ 2 & \text{on } y_1 = 0 \\ \tfrac{1}{2} & \text{on } y_2 = 0. \end{cases}$$

Also $\frac{dy_1}{dt} = 2y_1 + y_2 = \begin{cases} \tfrac{3}{2}y_1 \text{ on } y_2 = -\tfrac{1}{2}y_1, \text{ so incr. } y_1 \text{ if } y_1 > 0, \text{ dec } y_1 \text{ if } y_1 < 0 \\ y_2 \text{ on } y_1 = 0, \text{ so incr. } y_1 \text{ if } y_2 > 0, \text{ decr } y_1 \text{ if } y_2 < 0 \\ 2y_1 \text{ on } y_2 = 0, \text{ so incr. } y_1 \text{ if } y_1 > 0, \text{ decr. } y_1 \text{ if } y_1 < 0 \end{cases}$

$\frac{dy_2}{dt} = y_1 + 2y_2 = -3y_1$ on $y_2 = -2y_1$, so dec. y_2 if $y_1 > 0$, inc. y_2 if $y_1 < 0$.

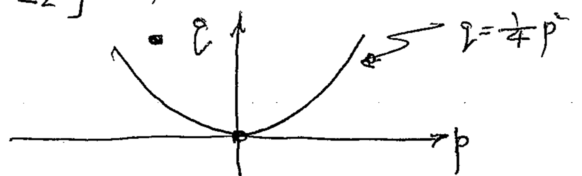
Putting all together:~

Also, if $\beta \neq 0$, $\underset{\sim}{y}(t) \approx \beta\binom{1}{-1}e^t$ as $t \to -\infty$.
$\therefore$ $y_2 \approx -y_1$ as $t \to -\infty$.



Similar to one in lectures!

K4.4#4* $\quad \underset{\sim}{y}' = A\underset{\sim}{y}, \quad A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \Rightarrow p = 2,\ q = 5,\ \Delta = 4 - 20 = -16$



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Attractive spiral. Stable

E'Values: $\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+2) + 5 = 0 \Rightarrow \lambda^2 + 2\lambda + 5 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

1

E'Vectors: $\lambda_1 = -1 + 2i$

$$\begin{bmatrix} 1-2i & 1 \\ -5 & -2-(-1+2i) \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} (1-2i)u + v = 0 \\ -5u + (-1-2i)v = 0 \end{matrix}\right\} \Rightarrow v = (2i-1)u$$

Watch out!

Choose $u = 1 \Rightarrow v = 2i - 1 \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 2i-1 \end{pmatrix}$

$\lambda_2 = -1 - 2i$ — just replace i by $-i$ everywhere. $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -2i-1 \end{pmatrix}$ 1

General solution: $\underset{\sim}{y}(t) = \alpha \begin{pmatrix} 1 \\ 2i-1 \end{pmatrix} e^{(-1+2i)t} + \beta \begin{pmatrix} 1 \\ -2i-1 \end{pmatrix} e^{(-1-2i)t}$

$$= e^{-t}\left[\alpha \begin{pmatrix} 1 \\ 2i-1 \end{pmatrix} e^{2it} + \beta \begin{pmatrix} 1 \\ -2i-1 \end{pmatrix} e^{-2it}\right]$$

complex form 1

$$= e^{-t}\left[\alpha \begin{pmatrix} 1 \\ 2i-1 \end{pmatrix}(\cos 2t + i \sin 2t) + \beta \begin{pmatrix} 1 \\ -2i-1 \end{pmatrix}(\cos 2t - i \sin 2t)\right]$$

$$= e^{-t}\begin{bmatrix} \alpha C + i\alpha S + \beta C - i\beta S \\ 2i\alpha C - \alpha C - 2\alpha S - i\alpha S - 2i\beta C - \beta C - 2\beta S + i\beta S \end{bmatrix}$$

$$= (\alpha + \beta) e^{-t} \begin{pmatrix} C \\ -C - 2S \end{pmatrix} + i(\alpha - \beta) e^{-t} \begin{pmatrix} S \\ 2C - S \end{pmatrix}$$

$C \equiv \cos 2t$
$S \equiv \sin 2t$

$$= c_1 e^{-t} \begin{pmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix}$$

Real form

1

2.6

See firstly that all trajectories approach origin as $t \to \infty$ (as $e^{-t} \to 0$). Next consider

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{-5y_1 - 2y_2}{y_2} = \begin{cases} 0 & \text{on } y_2 = -\frac{5}{2}y_1 \\ \pm\infty & \text{on } y_2 = 0 \\ -2 & \text{on } y_1 = 0 \quad (y_2 \neq 0) \end{cases}$$ (1)

Choose another convenient st. line – say $y_2 = y_1$

There $\frac{dy_2}{dy_1} = \frac{-7y_1}{y_1} = -7 \quad (y_1 \neq 0)$

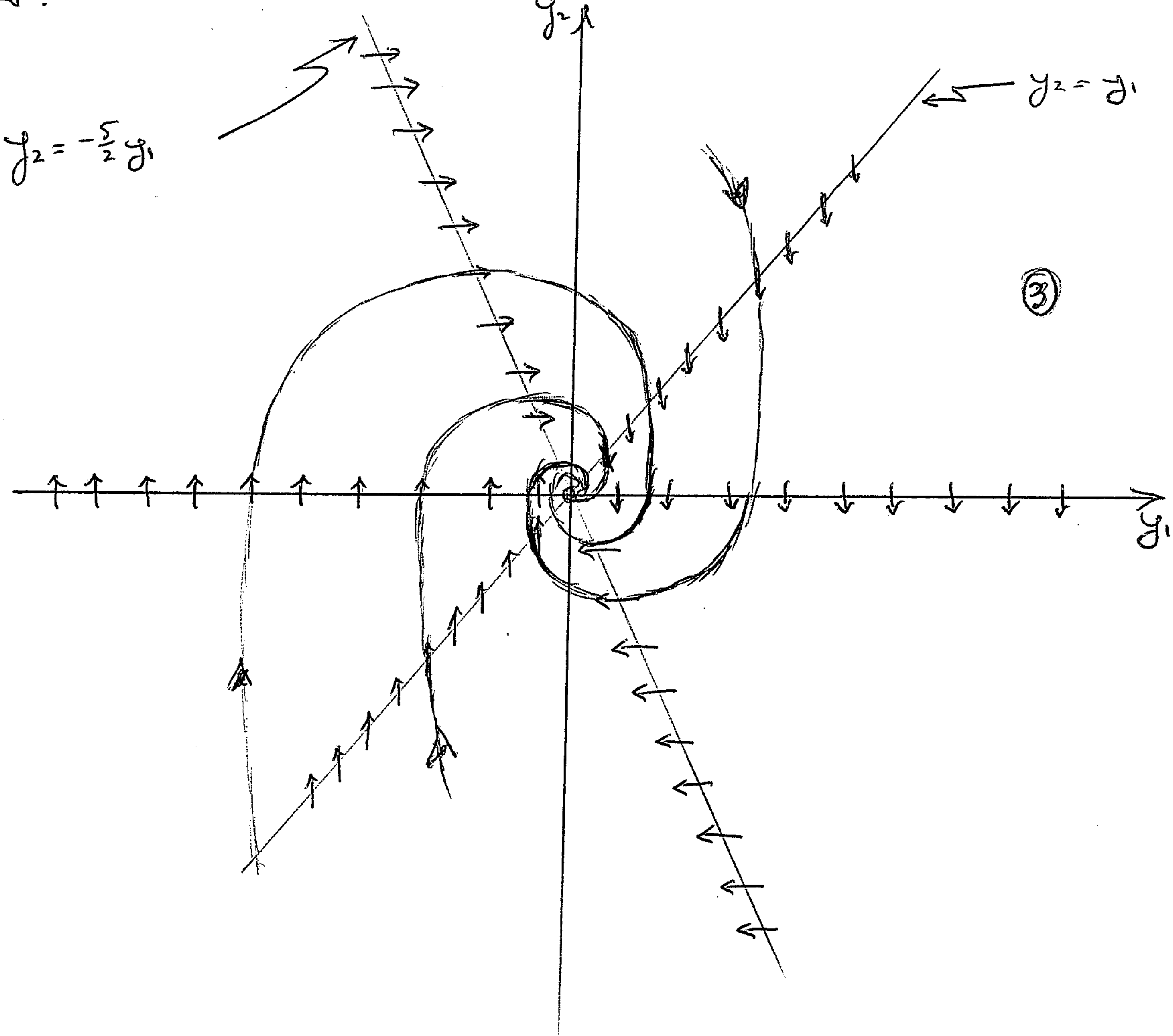
Also, $\frac{dy_1}{dt} = y_2$ so y_1 inc. wherever $y_2 > 0$ (1)

y_1 decr. wherever $y_2 < 0$

$\frac{dy_2}{dt} = -5y_1 - 2y_2 = -5y_1$ on $y_2 = 0$, so y_2 inc. where $y_1 < 0$

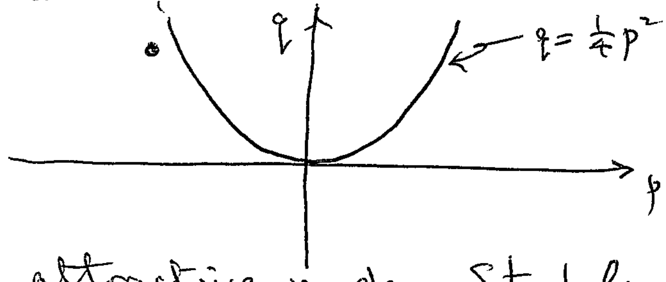
y_2 decr. where $y_1 > 0$.

So:-



(3)

K 4.4 #5 $\quad \underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \Rightarrow p = -8,\ q = 15,\ \Delta = 64 - 60 = 4$



Improper attractive node, Stable.

E'Values: $\begin{vmatrix} -4-\lambda & 1 \\ 1 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+4)^2 - 1 = 0 \Rightarrow \lambda^2 + 8\lambda + 15 = 0$

$\Rightarrow (\lambda+5)(\lambda+3) = 0$

$\Rightarrow \lambda_1 = -5,\ \lambda_2 = -3.$

E'Vectors: $\lambda_1 = -5$ $\quad \begin{bmatrix} -4+5 & 1 \\ 1 & -4+5 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} u+v=0 \\ u+v=0 \end{matrix}\right\} \Rightarrow v = -u$

Choose $u = 1 \Rightarrow v = -1 \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -3$ $\quad \begin{bmatrix} -4+3 & 1 \\ 1 & -4+3 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -u+v=0 \\ u-v=0 \end{matrix}\right\} \Rightarrow u = v$

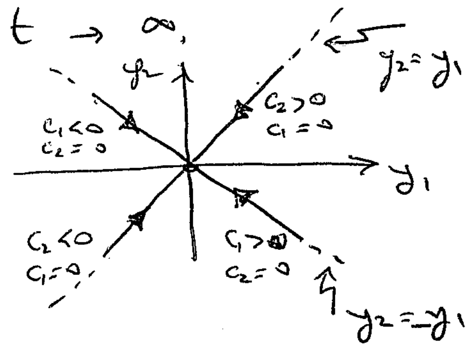
Choose $u = 1 \Rightarrow v = 1 \Rightarrow \underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution: $\quad \underset{\sim}{y}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-5t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}$

See all trajectories approach $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

If $c_2 = 0$, then $y_2 = -y_1 = -c_1 e^{-5t}$

So:



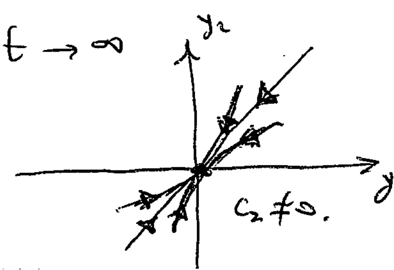
If $c_1 = 0$, then $y_2 = y_1 = c_2 e^{-3t}$

Also, as $t \to \infty$, $e^{-5t} << e^{-3t}$, so if $c_2 \neq 0$,

then $\underset{\sim}{y}(t) \approx c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}$ as $t \to \infty$

i.e. $y_2 \approx y_1 \approx c_2 e^{-3t}$ as $t \to \infty$

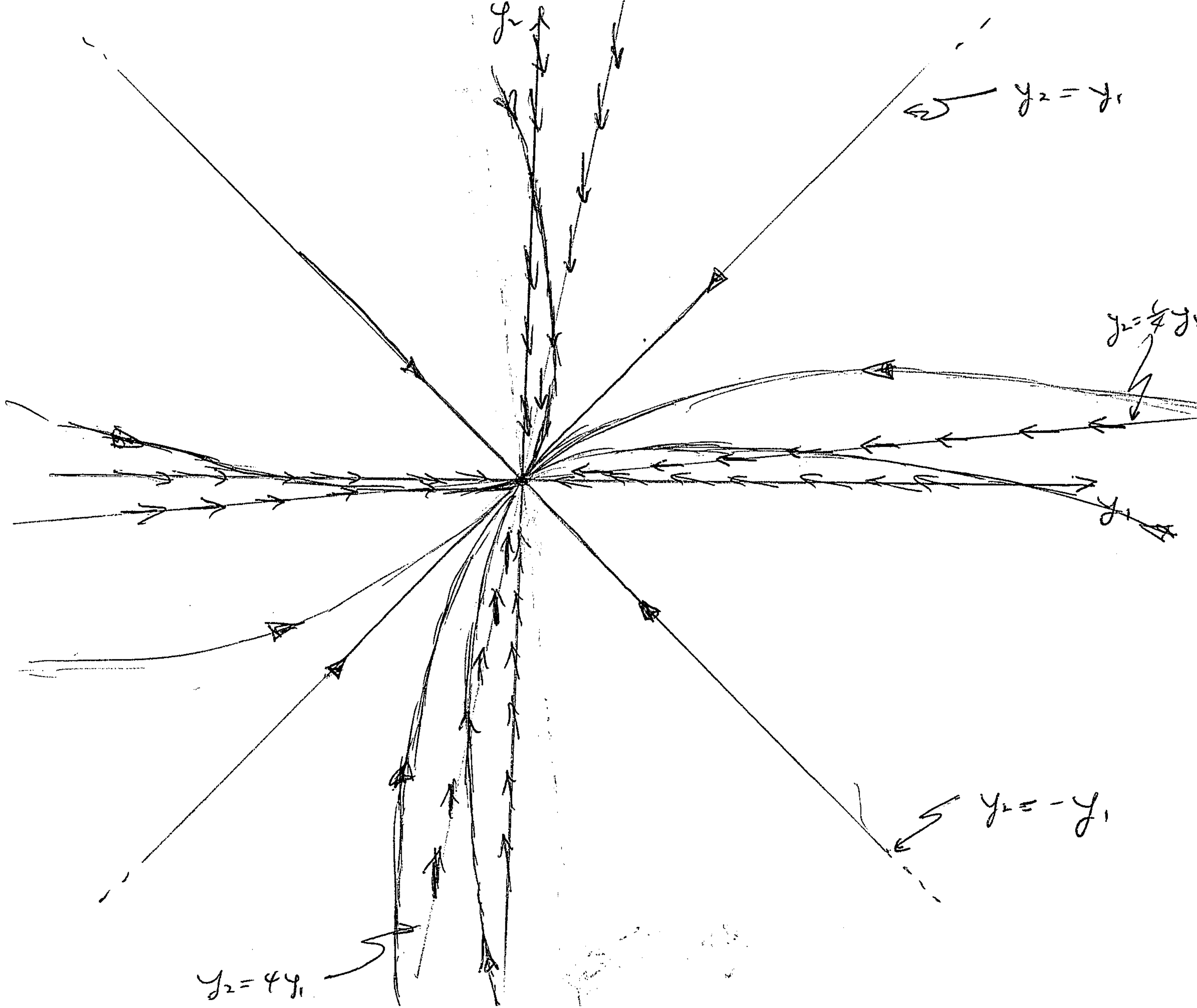
So:



Then: $$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{y_1 - 4y_2}{-4y_1 + y_2} = \begin{cases} 0 & \text{on } y_2 = \frac{1}{4}y_1 \\ \pm\infty & \text{on } y_2 = 4y_1 \\ -\frac{1}{4} & \text{on } y_2 = 0 \\ -4 & \text{on } y_1 = 0 \end{cases}$$

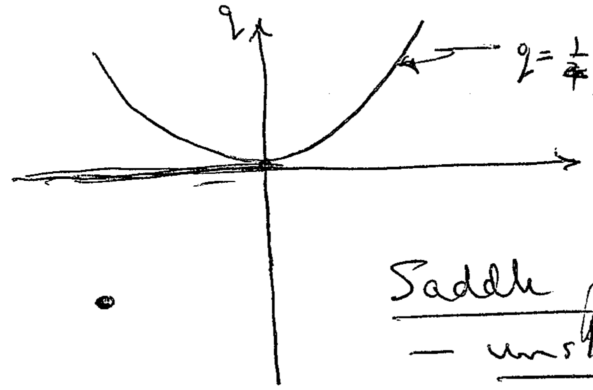
$$\frac{dy_1}{dt} = -4y_1 + y_2 = \begin{cases} -\frac{15}{4}y_1 & \text{on } y_2 = \frac{1}{4}y_1, \text{ so } \begin{array}{l} y_1 \text{ incr. if } y_1 < 0 \\ y_1 \text{ decr. if } y_1 > 0 \end{array} \\ -4y_1 & \text{on } y_2 = 0, \text{ so } \begin{array}{l} y_1 \text{ incr. if } y_1 < 0 \\ y_1 \text{ decr. if } y_1 > 0 \end{array} \\ y_2 & \text{on } y_1 = 0, \text{ so } \begin{array}{l} y_1 \text{ incr. if } y_2 > 0 \\ y_1 \text{ decr. if } y_2 < 0 \end{array} \end{cases}$$

$$\frac{dy_2}{dt} = y_1 - 4y_2 = -15y_1 \text{ on } y_2 = 4y_1, \text{ so } \begin{array}{l} y_2 \text{ incr. if } y_1 < 0 \\ y_2 \text{ decr. if } y_1 > 0. \end{array}$$



2.9

K4.4 #6 $\underset{\sim}{y}' = A\underset{\sim}{y}$, $A = \begin{bmatrix} 1 & 10 \\ 7 & -8 \end{bmatrix}$ $p = -7$, $q = -78$, $\Delta = 49 + 312 = 361$



Saddle point — unstable 1

E'Values: $\begin{vmatrix} 1-\lambda & 10 \\ 7 & -8-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda + 8) - 70 = 0$

$$\Rightarrow \lambda^2 + 7\lambda - 78 = 0$$

$$\Rightarrow \lambda = \frac{-7 \pm \sqrt{49 + 312}}{2} = \frac{-7 \pm 19}{2}$$

$\lambda_1 = 6$, $\lambda_2 = -13$. 1

E'Vectors: $\lambda_1 = 6$ $\begin{bmatrix} 1-6 & 10 \\ 7 & -8-6 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -5u + 10v = 0 \\ 7u - 14v = 0 \end{matrix}\right\} u = 2v.$

Choose $v = 1 \Rightarrow u = 2 \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ 1

$\lambda_2 = -13$ $\begin{bmatrix} 1+13 & 10 \\ 7 & -8+13 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 14u + 10v = 0 \\ 7u + 5v = 0 \end{matrix}\right\} \Rightarrow v = -\frac{7}{5}u$

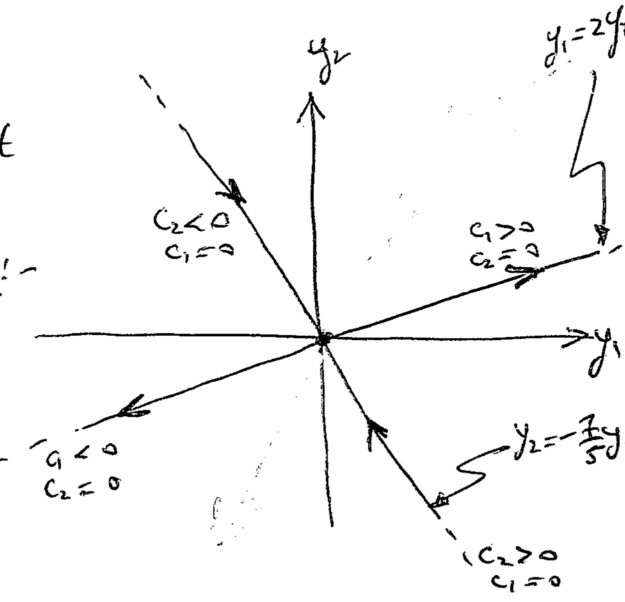
Choose $u = 5$, then $v = -7 \Rightarrow \underset{\sim}{x}^{(2)} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$

General solution: $\underset{\sim}{y}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 5 \\ -7 \end{pmatrix} e^{-13t}$ 1

If $c_2 = 0$, then $y_1 = 2y_2 = 2c_1 e^{6t}$

If $c_1 = 0$, then $y_2 = -\frac{7}{5}y_1 = -7c_2 e^{-13t}$

So:-



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Now consider $\dfrac{dy_2}{dy_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{7y_1 - 8y_2}{y_1 + 10y_2}$

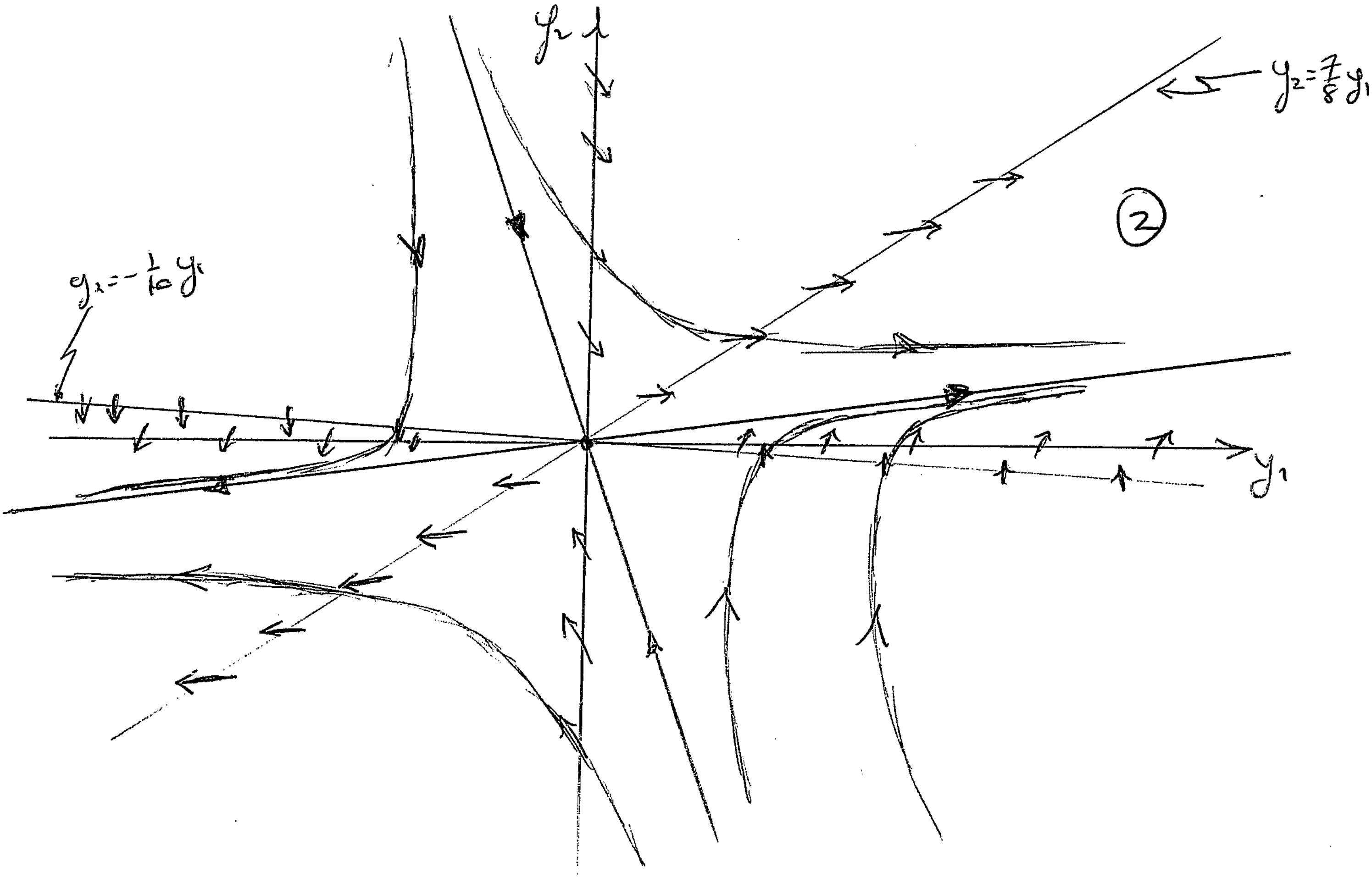
$$= \begin{cases} 0 & \text{on } y_2 = \frac{7}{8}y_1 \\ \pm\infty & \text{on } y_2 = -\frac{1}{10}y_1 \\ 7 & \text{on } y_2 = 0 \\ -\frac{4}{5} & \text{on } y_1 = 0 \end{cases} \quad (1)$$

Also $\dfrac{dy_1}{dt} = y_1 + 10y_2 = \dfrac{78}{8}y_1$ on $y_2 = \frac{7}{8}y_1$, so y_1 incr. if $y_1 > 0$
y_1 decr. if $y_1 < 0$

$\dfrac{dy_2}{dt} = 7y_1 - 8y_2 = \dfrac{78}{10}y_1$ on $y_2 = -\frac{1}{10}y_1$, so y_2 incr. if $y_1 > 0$
y_2 decr. if $y_1 < 0$.

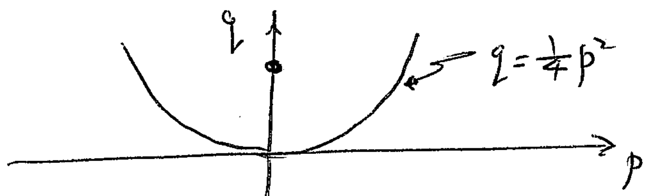
$\dfrac{dy_1}{dt} = y_1$ on $y_2 = 0$ so y_1 incr. where $y_1 > 0$
y_1 decr. where $y_1 < 0$. (1)

$\dfrac{dy_1}{dt} = 10y_2$ on $y_1 = 0$ so y_1 incr. where $y_2 > 0$
y_1 decr. where $y_2 < 0$.



K4.4 #7 $\quad \underset{\sim}{y}' = A\underset{\sim}{y}, \quad A = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \Rightarrow p = 0, \; q = 16$

$\Delta = 0 - 64$



Centre stable (but not attractive)

E' Values: $\begin{vmatrix} 0-\lambda & -2 \\ 8 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

$\lambda_1 = 4i, \quad \lambda_2 = -4i$

E' Vectors: $\lambda_1 = 4i$: $\begin{bmatrix} 0-4i & -2 \\ 8 & 0-4i \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -4iu - 2v = 0 \\ 8u - 4iv = 0 \end{matrix}\right\} \Rightarrow v = -2iu$

Choose $u = 1 \Rightarrow v = -2i \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$

$\lambda_2 = -4i \qquad i \leftrightarrow -i \qquad \underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$

General solution: $\underset{\sim}{y}(t) = c_1 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{4it} + c_2 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{-4it}$. ⊛

Complex form

$$= \begin{pmatrix} c_1(C + iS) \\ -2ic_1(C + iS) \end{pmatrix} + \begin{pmatrix} c_2(C - iS) \\ 2ic_2(C - iS) \end{pmatrix}$$

(where $C = \cos 4t$, $S = \sin 4t$)

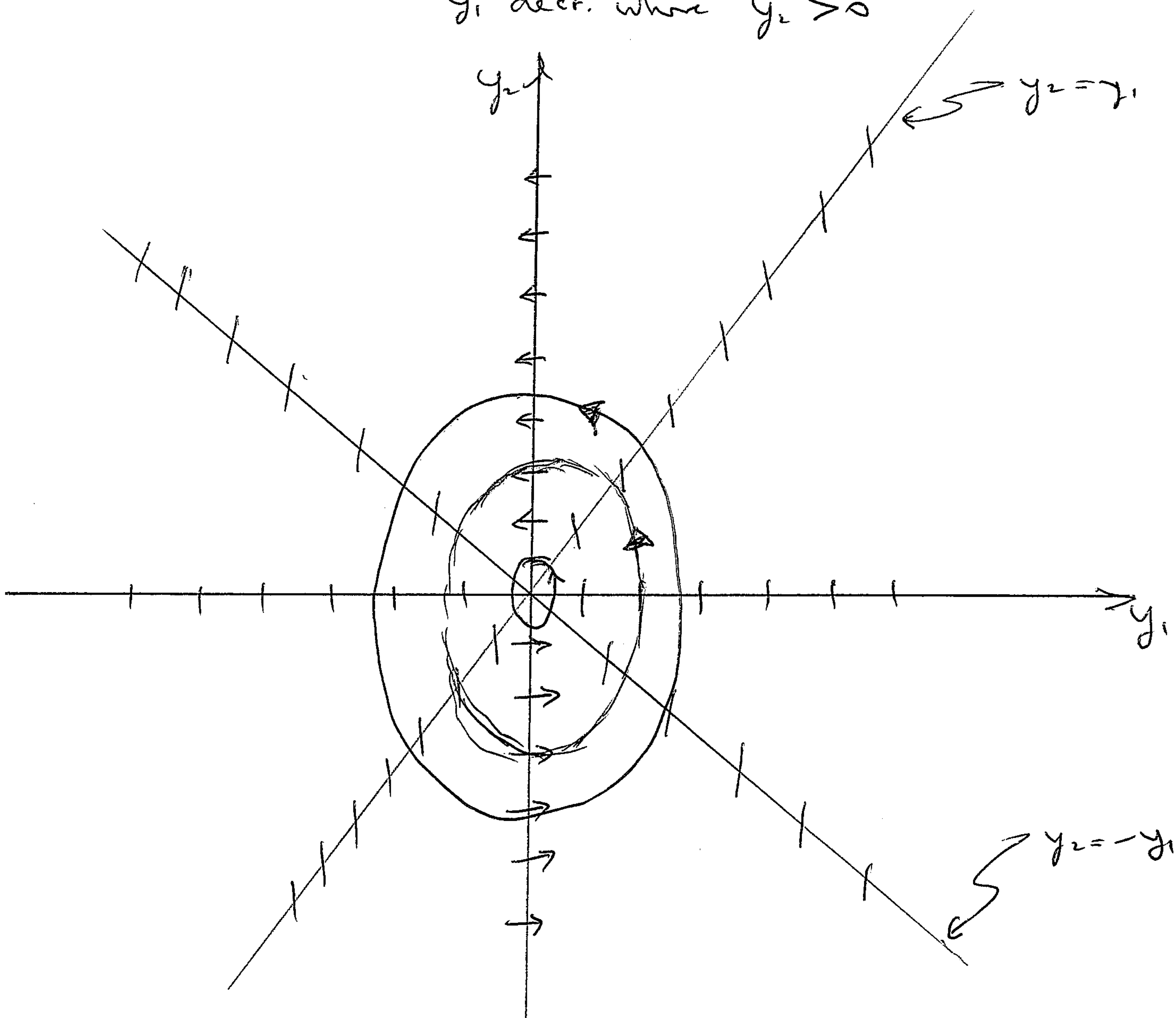
$$= (c_1 + c_2)\begin{pmatrix} C \\ 2S \end{pmatrix} + i(c_1 - c_2)\begin{pmatrix} S \\ -2C \end{pmatrix}$$

So real form of solution is

$$\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = d_1 \begin{pmatrix} \cos 4t \\ 2\sin 4t \end{pmatrix} + d_2 \begin{pmatrix} \sin 4t \\ -2\cos 4t \end{pmatrix}$$

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{8y_1}{-2y_2} = \begin{cases} 0 & \text{where } y_1 = 0 \\ \pm\infty & \text{where } y_2 = 0 \\ -4 & \text{where } y_1 = y_2 \\ +4 & \text{where } y_1 = -y_2 \end{cases}$$

$$\frac{dy_1}{dt} = -2y_2, \text{ so} \quad \begin{matrix} y_1 \text{ incr. where } y_2 < 0 \\ y_1 \text{ decr. where } y_2 > 0 \end{matrix}$$

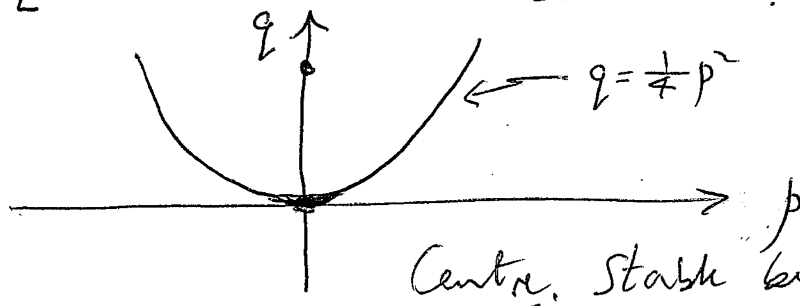


[Can get path equation for trajectories: From ⊛, $2iy_1 - y_2 = 4iC_1e^{4it}$
$2iy_1 + y_2 = 4iC_2e^{-4it}$

$$\Rightarrow (2iy_1 - y_2)(2iy_1 + y_2) = -16C_1C_2 \Rightarrow 4y_1^2 + y_2^2 = 16C_1C_2 = 4(d_1^2 + d_2^2)$$

– ellipse]

K4.4 #8 $\underset{\sim}{y}' = \begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix} \underset{\sim}{y} \Rightarrow p = 0,\ q = -9+25 = 16$
$\Delta = 0 - 64$ (1)



Centre. Stable but not attractive (1)

E'values: $\begin{vmatrix} 3-\lambda & 5 \\ -5 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-3)(\lambda+3)+25 = 0$

$\Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

E'values $\lambda_1 = 4i,\ \lambda_2 = -4i$ (1)

E'vectors: $\lambda_1 = 4i$ $\begin{bmatrix} 3-4i & 5 \\ -5 & -3-4i \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} (3-4i)u + 5v = 0 \\ -5u - (3+4i)v = 0 \end{matrix}\right\} \Rightarrow v = \left(\frac{-3+4i}{5}\right)u$

Choose $u = 5, \Rightarrow v = -3+4i \Rightarrow \underset{\sim}{x}^{(1)} = \begin{pmatrix} 5 \\ -3+4i \end{pmatrix}$

$\lambda_2 = -4i$ $i \longleftrightarrow -i \Rightarrow \underset{\sim}{x}^{(2)} = \begin{pmatrix} 5 \\ -3-4i \end{pmatrix}$ (1)

General solution: $\underset{\sim}{y}(t) = c_1 \begin{pmatrix} 5 \\ -3+4i \end{pmatrix} e^{4it} + c_2 \begin{pmatrix} 5 \\ -3-4i \end{pmatrix} e^{-4it}$ ⊛

complex form   (1)

$= \begin{pmatrix} 5c_1(C + iS) \\ (-3+4i)\,c_1\,(C+iS) \end{pmatrix} + \begin{pmatrix} 5c_2(C - iS) \\ (-3-4i)c_2(C-iS) \end{pmatrix}$ $C \equiv \cos 4t$, $S \equiv \sin 4t$

$= (c_1+c_2)\begin{pmatrix} 5C \\ -3C-4S \end{pmatrix} + i(c_1-c_2)\begin{pmatrix} 5S \\ -3S+4C \end{pmatrix}$

&

$\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = d_1 \begin{pmatrix} 5\cos 4t \\ -3\cos 4t - 4\sin 4t \end{pmatrix} + d_2 \begin{pmatrix} 5\sin 4t \\ -3\sin 4t + 4\cos 4t \end{pmatrix}$ (1)

Real form of general solution

$\begin{pmatrix} d_1 = c_1 + c_2 \\ d_2 = i(c_1 - c_2) \end{pmatrix}$