MATH2010/2100 Solutions 3 AOB

K4.5#2*

$y_1' = 4y_1 - y_1^2 = F_1(y_1, y_2)$

$y_2' = y_2 \quad = F_2(y_1, y_2)$

Critical points: $F_1 = 0$ and $F_2 = 0$, so

$y_1(4 - y_1) = 0$ and $y_2 = 0$

$\Rightarrow$ $y_1 = 0$ or 4 and $y_2 = 0$.

$\Rightarrow$ critical points $P_1 = (0, 0)$, $P_2 = (4, 0)$

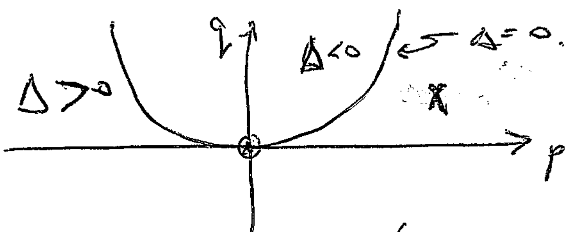
Now $\dfrac{\partial F_1}{\partial y_1} = 4 - 2y_1$, $\dfrac{\partial F_1}{\partial y_2} = 0$, $\dfrac{\partial F_2}{\partial y_1} = 0$, $\dfrac{\partial F_2}{\partial y_2} = 1$

So Jacobian matrix $F = \begin{bmatrix} 4-2y_1 & 0 \\ 0 & 1 \end{bmatrix}$

First critical pt. $(0,0)$

$F = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow p = 5,\ q = 4$

$\Delta = p^2 - 4q = 9.$



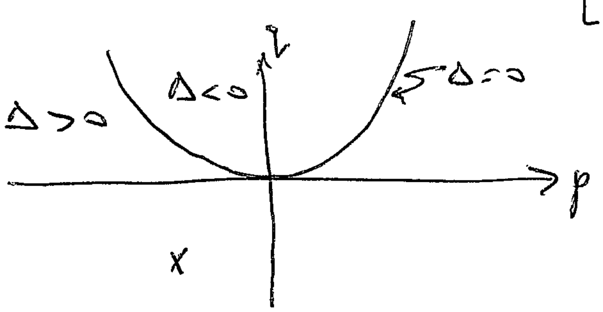
Improper node
Repulsive
Unstable

[Linearized equations are $\underset{\sim}{Y}' = F\underset{\sim}{Y}$, $Y_1 = y_1 - 0 = y_1$
$Y_2 = y_2 - 0 = y_2$.]

Second critical pt. $(4, 0)$

$F = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow p = -3,\ q = -4$

$\Delta = 9 + 36 = 45.$



Saddle pt.
Unstable.

[Linearized equations are $\underset{\sim}{Y}' = F\underset{\sim}{Y}$, $Y_1 = y_1 - 4$, $Y_2 = y_2 - 0 = y_2$]

K4.5#3 $y_1' = 4y_2 = F_1(y_1, y_2)$; $y_2' = 2y_1 - y_1^2 = F_2(y_1, y_2)$

Critical points: $F_1 = 0$ and $F_2 = 0$

$\Rightarrow 4y_2 = 0$ and $y_1(2 - y_1) = 0$

$\Rightarrow y_2 = 0$ and $y_1 = 0$ or 2

$\Rightarrow$ Critical points $(0,0)$ and $(2,0)$

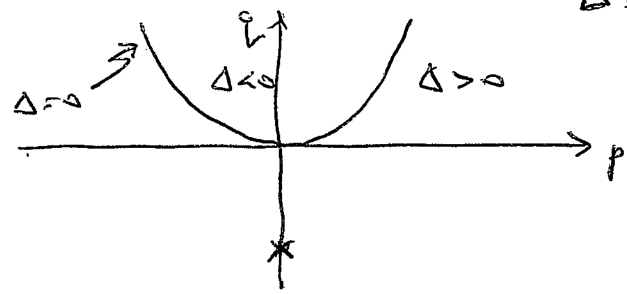
$$\frac{\partial F_1}{\partial y_1} = 0, \quad \frac{\partial F_1}{\partial y_2} = 4, \quad \frac{\partial F_2}{\partial y_1} = 2 - 2y_1, \quad \frac{\partial F_2}{\partial y_2} = 0$$

Jacobian matrix $F = \begin{bmatrix} 0 & 4 \\ 2-2y_1 & 0 \end{bmatrix}$

Near $(0,0)$: $\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$, $F = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} \Rightarrow p = 0, \; q = -8$, $\Delta = 32$

$Y_1 = y_1 - 0 = y_1$

$Y_2 = y_2 - 0 = y_2$



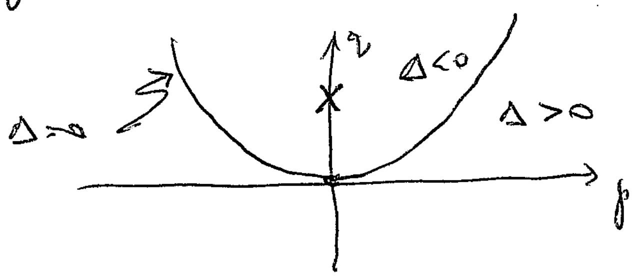
Saddle pt
unstable

Near $(2,0)$: $\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$

$Y_1 = y_1 - 2$,

$Y_2 = y_2 - 0 = y_2$.

$F \approx \begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix} \Rightarrow p = 0, \; q = 8$, $\Delta = -32$



Centre

Stable but not attractive

K4.5 #4

$$y_1' = -3y_1 + y_2 - y_2^2 = F_1(y_1, y_2)$$
$$y_2' = y_1 - 3y_2 = F_2(y_1, y_2)$$

Critical pts: $F_1 = 0$ and $F_2 = 0$.

$$\Rightarrow \quad -3y_1 + y_2 - y_2^2 = 0 \quad \text{and} \quad y_1 - 3y_2 = 0.$$
$$\hookrightarrow \quad y_1 = 3y_2$$

$$-9y_2 + y_2 - y_2^2 = 0$$
$$\hookrightarrow \quad y_2(y_2 + 8) = 0.$$
$$\hookrightarrow \quad y_2 = 0 \text{ or } -8$$

Critical pts.

$(0, 0)$, $(-24, -8)$

$$F_{11} = \frac{\partial F_1}{\partial y_1} = -3, \quad F_{12} = \frac{\partial F_1}{\partial y_2} = 1 - 2y_2$$
$$F_{21} = \frac{\partial F_2}{\partial y_1} = 1, \quad F_{22} = \frac{\partial F_2}{\partial y_2} = -3.$$

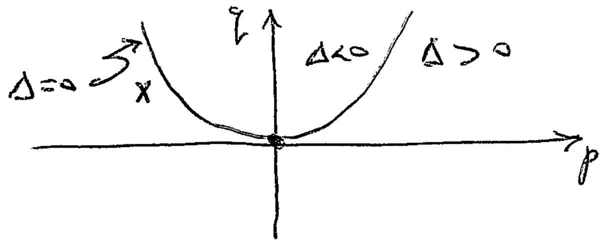
$$\Rightarrow \text{Jacobian matrix} \quad F = \begin{bmatrix} -3 & 1-2y_2 \\ 1 & -3 \end{bmatrix}$$

Near $(0,0)$: $\underline{Y}' \approx F\underline{Y}$, $F = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow p = -6,\ q = 8$

$Y_1 = y_1 - 0 = y_1$

$Y_2 = y_2 - 0 = y_2$

$\Delta = 36 - 32 = 4.$



Improper node

Attractive

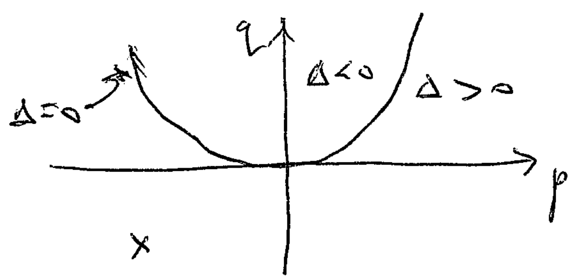
stable

Near $(-24,-8)$: $\underline{Y}' \approx F\underline{Y}$, $F = \begin{bmatrix} -3 & 17 \\ 1 & -3 \end{bmatrix} \Rightarrow p = -6,\ q = -8$

$Y_1 = y_1 + 24$

$Y_2 = y_2 + 8$

$\Delta = 36 + 32 = 68$



Saddle point

unstable

K4.5 #5

$$y_1' = -y_1 + y_2 - y_2^2 = F_1(y_1, y_2)$$
$$y_2' = -y_1 - y_2 = F_2(y_1, y_2)$$

Critical points: $F_1 = 0$ and $F_2 = 0$

$\hookrightarrow -y_1 + y_2 - y_2^2 = 0$ and $-y_1 - y_2 = 0$

$\hookrightarrow y_2 = -y_1$

$$y_2 + y_2 - y_2^2 = 0$$
$$y_2(2 - y_2) = 0$$
$$y_2 = 0 \text{ or } 2$$

Critical pts $(0,0)$ and $(-2,2)$

$$F_{11} = \frac{\partial F_1}{\partial y_1} = -1, \quad F_{12} = \frac{\partial F_1}{\partial y_2} = 1 - 2y_2, \quad F_{21} = \frac{\partial F_2}{\partial y_1} = -1, \quad F_{22} = \frac{\partial F_2}{\partial y_2} = -1$$

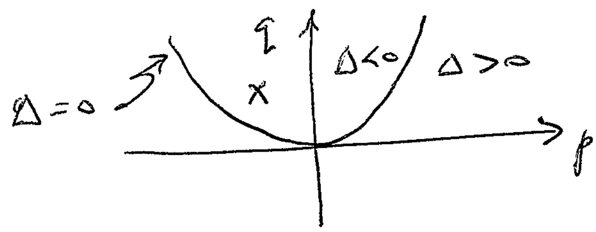
Jacobian matrix: $F = \begin{bmatrix} -1 & 1-2y_2 \\ -1 & -1 \end{bmatrix}$

Near $(0,0)$:

$Y_1 = y_1 - 0 = y_1$
$Y_2 = y_2 - 0 = y_2$

$\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$, $F = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow p = -2,\ q = 2$

$\Delta = 4 - 8 = -4.$



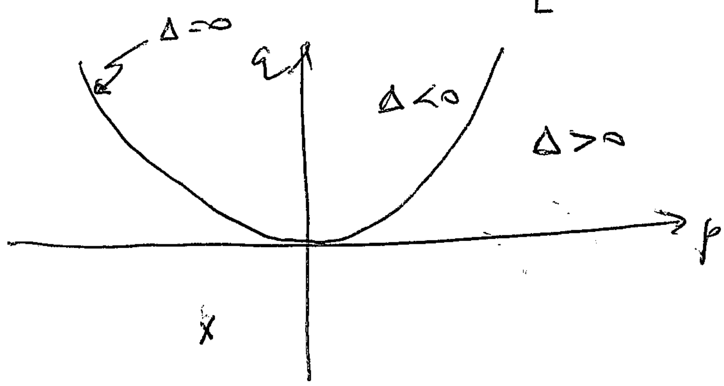
Spiral
Attractive
Stable.

Near $(-2,2)$:

$Y_1 = y_1 + 2$
$Y_2 = y_2 - 2$

$\underset{\sim}{Y}' \approx F\underset{\sim}{Y}$, $F = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix} \Rightarrow p = -2,\ q = -3$

$\Delta = 4 + 12 = 16.$



Saddle pt
Unstable.

3.5

K4.5 #6 $y_1' = y_2 - y_2^2 = F_1(y_1, y_2) \qquad y_2' = y_1 - y_1^2 = F_2(y_1, y_2)$

Critical points where $F_1 = 0$ and $F_2 = 0$.

$\hookrightarrow \quad y_2(1-y_2) = 0$ and $y_1(1-y_1) = 0$

$\hookrightarrow \quad y_2 = 0$ or 1 and $y_1 = 0$ or 1

So critical points are $(0,0), (0,1), (1,0), (1,1)$.

$$F_{11} = \frac{\partial F_1}{\partial y_1} = 0, \quad F_{12} = \frac{\partial F_1}{\partial y_2} = 1 - 2y_2, \quad F_{21} = \frac{\partial F_2}{\partial y_1} = 1 - 2y_1, \quad F_{22} = \frac{\partial F_2}{\partial y_2} = 0$$

Jacobian matrix: $F = \begin{bmatrix} 0 & 1-2y_2 \\ 1-2y_1 & 0 \end{bmatrix}$

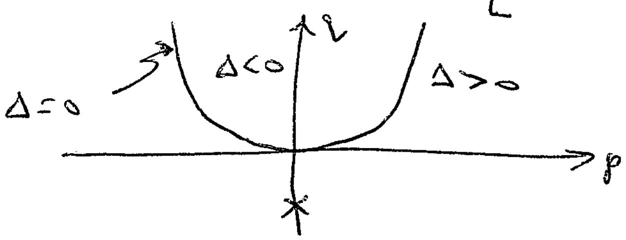
Near $(0,0)$:

$Y_1 = y_1 - 0 = y_1$

$Y_2 = y_2 - 0 = y_2$

$\underline{Y}' \approx F\underline{Y}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow p = 0, q = -1$

$\Delta = 0 + 4 = 4.$



Saddle point

unstable

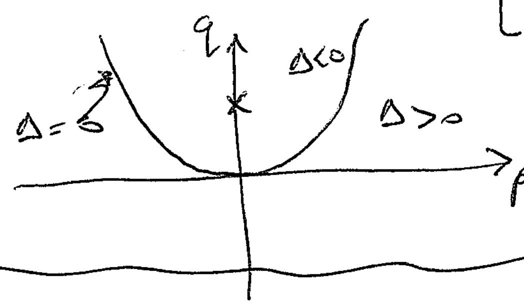
Near $(0,1)$:

$Y_1 = y_1 - 0 = y_1$

$Y_2 = y_2 - 1$

$\underline{Y}' \approx F\underline{Y}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow p = 0, \quad q = 1$

$\Delta = 0 - 4 = -4$



Centre

Stable

Not attractive

Near $(1,0)$:

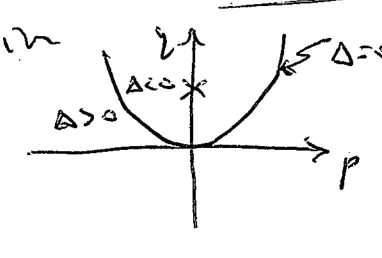
$Y_1 = y_1 - 1$

$Y_2 = y_2 - 0 = y_2$

$\underline{Y}' \approx F\underline{Y}, \quad F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} p = 0, \quad q = 1 \\ \Delta = -4 \end{array}\right\}$ Same as

— again centre, stable, not attractive

[Trajectories around these two center go in opposite directions — can you see it?]

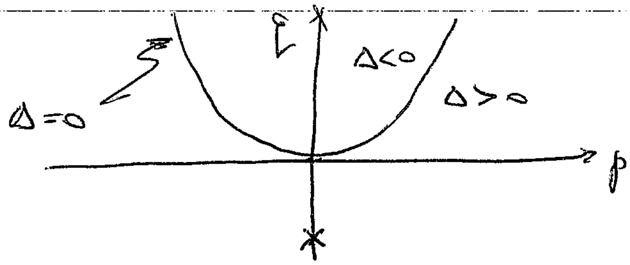


<u>New (1,1):</u>

$Y_1 = y_1 - 1$
$Y_2 = y_2 - 1$

$$\underset{\sim}{Y}' \approx F \underset{\sim}{Y}, \quad F = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \Rightarrow p = 0,\ q = -1$$

$$\Delta = 0 + 4 = 4$$



(as for (0,0))

Saddle pt.

<u>Unstable</u>

K4.5 #8* $\quad y'' + 9y + y^2 = 0$

Put $y_1 = y$, $y_2 = y'$, then

$$y_1' = y' = y_2, \quad y_2' = y'' = -9y - y^2 = -9y_1 - y_1^2$$

i.e.
$$\begin{cases} y_1' = y_2 = F_1(y_1, y_2) \\ y_2' = -9y_1 - y_1^2 = F_2(y_1, y_2) \end{cases} \qquad \begin{aligned} F_1 &= y_2 \\ F_2 &= -9y_1 - y_1^2 \end{aligned}$$

Critical points:- $F_1 = 0$ and $F_2 = 0$

$$\Rightarrow \quad y_2 = 0 \quad \text{and} \quad -y_1(y_1 + 9) = 0$$

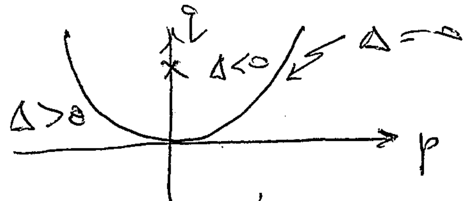
So $(0,0)$ and $(-9, 0)$ are critical points.

Jacobian matrix:- $\dfrac{\partial F_1}{\partial y_1} = 0 \qquad \dfrac{\partial F_1}{\partial y_2} = 1$

$\dfrac{\partial F_2}{\partial y_1} = -9 - 2y_1 \qquad \dfrac{\partial F_2}{\partial y_2} = 0$

$$F = \begin{bmatrix} 0 & 1 \\ -9-2y_1 & 0 \end{bmatrix}$$

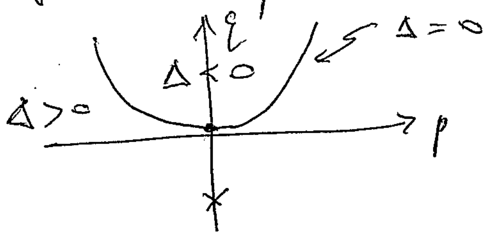
Critical point $(0,0)$:- $\quad F = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \Rightarrow p = 0, \; q = 9$
$\Delta = 0 - 4 \cdot 9 = -36$



Centre $\quad$ Stable but not attractive.

Linearized equations are $\underset{\sim}{Y}' = F\underset{\sim}{Y}$, $Y_1 = y_1 - 0$, $Y_2 = y_2 - 0$

Critical point $(-9, 0)$:- $\quad F = \begin{bmatrix} 0 & 1 \\ +9 & 0 \end{bmatrix} \Rightarrow p = 0, \; q = -9$
$\Delta = 0 - 4(-9) = +36$



Saddle $\quad$ Unstable

Linearized equations are $\underset{\sim}{Y}' = F\underset{\sim}{Y}$, $Y_1 = y_1 + 9$, $Y_2 = 0$