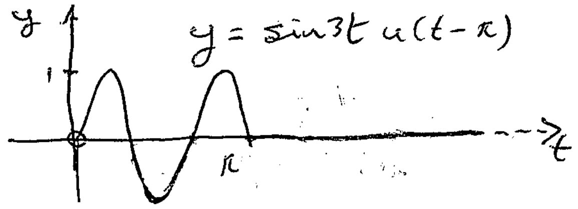
MATH2010 Solutions 5 2006/1 AJB

K6.3 #4

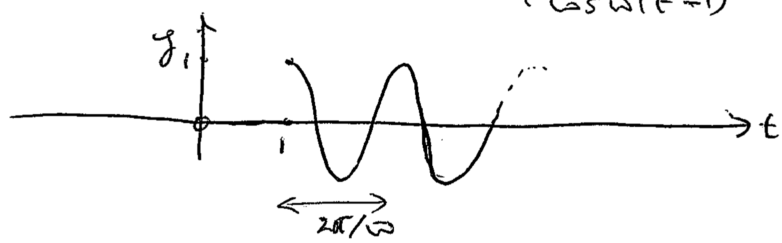


$$f(t) = \begin{cases} \sin(3t) & 0<t<\pi \\ 0 & t>\pi \end{cases}$$
$$= \sin(3t)\left[1-u(t-\pi)\right]$$
$$= \sin(3t) + \sin[3(t-\pi)]\,u(t-\pi)$$

$\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$, so $\mathcal{L}(f(t)) = \frac{3}{s^2+9}\left(1+e^{-\pi s}\right)$

K6.3 #14 $F(s) = e^{-s}\left[\frac{s}{s^2+\omega^2}\right]$. $\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos(\omega t)$, so

$$f(t) = \cos[\omega(t-1)]\,u(t-1) = \begin{cases} 0 & 0<t<1 \\ \cos\omega(t-1) & t>1 \end{cases}$$



K6.3 #24 $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$

Take L.T. of both sides of ODE, setting $\mathcal{L}(y(t)) = Y(s)$

$$9\left[s^2Y(s) - sy(0) - y'(0)\right] - 6\left[sY(s) - y(0)\right] + Y(s) = 0$$
$$9\left[s^2Y(s) - 3s - 1\right] - 6\left[sY(s) - 3\right] + Y(s) = 0$$
$$(9s^2 - 6s + 1)Y(s) = 27s + 9 - 18 = 27s - 9$$
$$\Rightarrow \quad Y(s) = \frac{27s-9}{(3s-1)^2} = \frac{9(3s-1)}{(3s-1)^2} = \frac{3}{s-\frac{1}{3}}$$
$$\Rightarrow \quad y(t) = 3e^{t/3}.$$

K6.3 #28 $y'' + 3y' + 2y = r(t) = \begin{cases} 1 & 0<t<1 \\ 0 & t>1 \end{cases} = 1 - u(t-1)$

$y(0) = 0$, $y'(0) = 0$

Let $\mathcal{L}(y(t)) = Y(s)$. Then $\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$

$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

Also $\mathcal{L}(r(t)) = \frac{1}{s} - \frac{e^{-s}}{s}$

Then, taking L.T. of both sides of O.D.E, we get

$$(s^2+3s+2)Y(s) = \frac{1-e^{-s}}{s}$$
$$\text{so} \quad Y(s) = (1-e^{-s})\left[\frac{1}{s(s+1)(s+2)}\right]$$

Now $\dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$\Rightarrow \quad 1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

$s=0$: $\quad 1 = 2A + 0 + 0 \quad \Rightarrow \quad A = \frac{1}{2}$

$s=-1$: $\quad 1 = 0 - B + 0 \quad \Rightarrow \quad B = -1$

$s=-2$: $\quad 1 = 0 + 0 + 2C \quad \Rightarrow \quad C = \frac{1}{2}$

So $\dfrac{1}{s(s+1)(s+2)} = \dfrac{1/2}{s} - \dfrac{1}{s+1} + \dfrac{1/2}{s+2} = F(s)$, say

Then $Y(s) = F(s) - e^{-s}F(s)$

$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

$\mathcal{L}^{-1}(e^{-s}F(s)) = \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]u(t-1)$

So $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - \left[\frac{1}{2} - e\,e^{-t} + \frac{1}{2}e^{2}e^{-2t}\right]u(t-1)$

$$= \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} & 0 < t < 1 \\ (e-1)e^{-t} + \frac{1}{2}(1-e^{2})e^{-2t}, & t > 1 \end{cases}$$

K 6.5 #18 $\quad y'' + y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$

Let $Y(s) = \mathcal{L}(y(t))$. Then $\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$

$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

Also $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

So, taking L.T. of b.s. of O.D.E gives

(3)

$(s^2+1)Y(s) = \dfrac{1}{s^2+1}$

$\Rightarrow \quad Y(s) = \dfrac{1}{(s^2+1)^2} = F(s)^2$, where $F(s) = \dfrac{1}{s^2+1}$

Then $y(t) = f(t) * f(t)$ where $f(t) = \mathcal{L}^{-1}(F(s)) = \sin t$

Then $y(t) = \displaystyle\int_0^t \sin(\tau)\sin(t-\tau)\,d\tau$ (1)

Now $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

So $\sin(\tau)\sin(t-\tau) = \frac{1}{2}\cos(2\tau - t) - \frac{1}{2}\cos t$

So $y(t) = \frac{1}{2}\displaystyle\int_0^t \cos(2\tau - t)\,d\tau - \frac{1}{2}\cos t \int_0^t d\tau$

$= \frac{1}{4}\left[\sin(2\tau - t)\right]_0^t - \frac{1}{2}t\cos t$

$= \frac{1}{4}\sin t - \frac{1}{4}\sin(-t) - \frac{1}{2}t\cos t$

$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$ (3)

(S.3)

K6.7 #14 $y_1'(t) = -y_2(t)$, $y_2'(t) = -y_1(t) + 2[1-u(t-2\pi)]\cos t$;

$y_1(0) = 1$, $y_2(0) = 0$

Let $Y_1(s) = \mathcal{L}(y_1(t))$, $Y_2(s) = \mathcal{L}(y_2(t))$

Then $\mathcal{L}(y_1'(t)) = sY_1(s) - y_1(0) = sY_1(s) - 1$

$\mathcal{L}(y_2'(t)) = sY_2(s) - y_2(0) = sY_2(s)$

Also, $2[1-u(t-2\pi)]\cos t = 2\cos t - 2u(t-2\pi)\cos(t-2\pi)$

So $\mathcal{L}\left(2[1-u(t-2\pi)]\cos t\right) = \frac{2s}{s^2+1} - 2e^{-2\pi s}\frac{s}{s^2+1}$

Then, taking L.T. of b.s. of the two O.D.Es, we get

$sY_1 - 1 + Y_2 = 0$, $\quad sY_2 + Y_1 = (1-e^{-2\pi s})\frac{2s}{s^2+1}$ (3)

$\Rightarrow sY_2 = -s^2Y_1 + s \longrightarrow (1-s^2)Y_1 + s = (1-e^{-2\pi s})\frac{2s}{s^2+1}$

$\Rightarrow Y_1 = \frac{s(s^2+1) - 2s + 2se^{-2\pi s}}{(s^2-1)(s^2+1)} = \frac{s^3 - s + 2se^{-2\pi s}}{(s^2-1)(s^2+1)}$

$= \frac{s(s^2-1) + 2se^{-2\pi s}}{(s^2-1)(s^2+1)} = \frac{s}{s^2+1} + 2e^{-2\pi s}\frac{s}{(s^2+1)(s^2-1)}$

Now $\frac{s}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1}$

$\Rightarrow s = (As+B)(s^2-1) + C(s^2+1)(s+1) + D(s^2+1)(s-1)$

$\underline{s=1}$: $1 = 0 + 4C + 0 \Rightarrow C = 1/4$

$\underline{s=-1}$: $-1 = 0 + 0 - 4D \Rightarrow D = 1/4$

Coeff. of $\underline{s^3}$: $0 = A + C + D \Rightarrow A = -\frac{1}{2}$

Coeff. of $\underline{s^2}$: $0 = B + C - D \Rightarrow B = 0$

So $\frac{s}{(s^2+1)(s^2-1)} = \frac{-\frac{1}{2}s}{s^2+1} + \frac{1/4}{s-1} + \frac{1/4}{s+1}$

So $Y_1 = \frac{s}{s^2+1} + e^{-2\pi s}\left[\frac{-s}{s^2+1} + \frac{1/2}{s-1} + \frac{1/2}{s+1}\right]$ (4)

$\Rightarrow y_1(t) = \cos t + u(t-2\pi)\left[-\cos(t-2\pi) + \frac{1}{2}e^{t-2\pi} + \frac{1}{2}e^{-(t-2\pi)}\right]$

$= \cos t + u(t-2\pi)\left[-\cos t + \cosh(t-2\pi)\right]$ (2)

Also, from $sY_1 - 1 + Y_2 = 0$ (above) we have $Y_2 = 1 - sY_1$

$= 1 - \frac{s^2}{s^2+1} + e^{-2\pi s}\left[\frac{s^2}{s^2+1} - \frac{\frac{1}{2}s}{s-1} - \frac{\frac{1}{2}s}{s+1}\right]$

$= \frac{1}{s^2+1} + e^{-2\pi s}\left[\frac{s^2+1-1}{s^2+1} - \frac{\frac{1}{2}(s-1)+\frac{1}{2}}{s-1} - \frac{\frac{1}{2}(s+1)-\frac{1}{2}}{s+1}\right]$

$$= \frac{1}{s^2+1} + e^{-2\pi s}\left[1 - \frac{1}{s^2+1} - \frac{1}{2} - \frac{1/2}{s-1} - \frac{1}{2} + \frac{1/2}{s+1}\right]$$

$$= \frac{1}{s^2+1} + e^{-2\pi s}\left[-\frac{1}{s^2+1} - \frac{1/2}{s-1} + \frac{1/2}{s+1}\right] \qquad (2)$$

$$\Rightarrow \quad y_2(t) = \sin t + u(t-2\pi)\left[-\sin(t-2\pi) - \tfrac{1}{2}e^{t-2\pi} + \tfrac{1}{2}e^{-(t-2\pi)}\right]$$

$$= \sin t + u(t-2\pi)\left[-\sin t - \frac{e^{t-2\pi} - e^{-(t-2\pi)}}{2}\right]$$

$$= \sin t + u(t-2\pi)\left[-\sin t - \sinh(t-2\pi)\right] \qquad (2)$$

So

$$y_1(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ \cos t - \cos t + \cosh(t-2\pi) & t > 2\pi. \end{cases}$$

$$= \begin{cases} \cos t & 0 < t < 2\pi \\ \cosh(t-2\pi) & t > 2\pi. \end{cases}$$

$$y_2(t) = \begin{cases} \sin t & 0 < t < 2\pi \\ \sin t - \sin t - \sinh(t-2\pi) & t > 2\pi \end{cases}$$

$$= \begin{cases} \sin t & 0 < t < 2\pi \\ -\sinh(t-2\pi) & t > 2\pi. \end{cases}$$

5.5

K6-5 # 20*. $y'' + 5y' + 4y = 2e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

Let $Y(s) = \mathcal{L}(y(t))$. Then $\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$

and $\mathcal{L}(y''(t)) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s)$

Also $\mathcal{L}(2e^{-2t}) = \frac{2}{s+2}$, so, taking L.T. of b.s. of ODE

gives $(s^2 + 5s + 4) Y(s) = \frac{2}{s+2}$. or $(s+1)(s+4) Y(s) = \frac{2}{s+2}$ 2

$\Rightarrow Y(s) = \frac{2}{(s+1)(s+2)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+4}$, say

Then $2 = A(s+2)(s+4) + B(s+1)(s+4) + C(s+1)(s+2)$

$s = -1 \Rightarrow \quad 2 = 3A + 0 + 0 \Rightarrow \quad A = \frac{2}{3}$.

$s = -2 \Rightarrow \quad 2 = 0 - 2B + 0 \Rightarrow \quad B = -1$

$s = -4 \Rightarrow \quad 2 = 0 + 0 + 6C \Rightarrow \quad C = \frac{1}{3}$.

So $Y(s) = \frac{2/3}{s+1} - \frac{1}{s+2} + \frac{1/3}{s+4}$ 2

$\Rightarrow \quad y(t) = \frac{2}{3} e^{-t} - e^{-2t} + \frac{1}{3} e^{-4t}$. 2

[OR using convolution theorem:- OR

$Y(s) = \left[\frac{1}{(s+1)(s+4)}\right] \left(\frac{2}{s+2}\right)$

$U(s) = \frac{1}{(s+1)(s+4)} = \frac{E}{s+1} + \frac{F}{s+4} \Rightarrow \quad E = \frac{1}{3}, \ F = -\frac{1}{3}$.

Then $y(t) = u(t) * v(t)$ where $U(s) = \frac{1/3}{s+1} - \frac{\frac{1}{3}}{s+4}$ 2

$\left[\Rightarrow u(t) = \frac{1}{3}(e^{-t} - e^{-4t})\right]$

and $V(s) = \frac{2}{s+2} \quad (\Leftrightarrow v(t) = 2e^{-2t})$ 2

Then $y(t) = \frac{2}{3} \int_0^t e^{-2(t-\tau)} \left[e^{-\tau} - e^{-4\tau}\right] d\tau$

$= \frac{2}{3} e^{-2t} \int_0^t \left[e^{\tau} - e^{-2\tau}\right] d\tau$

$= \frac{2}{3} e^{-2t} \left[e^{\tau} + \frac{1}{2} e^{-2\tau}\right]_{\tau=0}^{\tau=t}$

$= \frac{2}{3} e^{-2t} \left[e^{t} + \frac{1}{2} e^{-2t} - \frac{3}{2}\right]$ 2

$= \frac{2}{3} e^{-t} - e^{-2t} + \frac{1}{3} e^{-4t}$]

K6.7 #12* $y_1' = 2y_1 + y_2 \qquad y_2' = 4y_1 + 2y_2 + 64t\,u(t-1)$

$y_1(0) = 2, \quad y_2(0) = 0.$

Let $Y_1(s) = \mathcal{L}(y_1(t))$ $[\Rightarrow \mathcal{L}(y_1'(t)) = sY_1(s) - y_1(0) = sY_1(s) - 2]$

& $Y_2(s) = \mathcal{L}(y_2(t))$ $[\Rightarrow \mathcal{L}(y_2'(t)) = sY_2(s) - y_2(0) = sY_2(s)]$

~~The~~ Also, $\mathcal{L}(64t\,u(t-1)) = \mathcal{L}[64(t-1)u(t-1) + 64u(t-1)]$

Since $\mathcal{L}(64t) = \frac{64}{s^2}$, and $\mathcal{L}(64) = \frac{64}{s}$

we get from 2nd Shifting Theorem

$$\mathcal{L}(64t\,u(t-1)) = 64e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right] \qquad 2$$

Now, taking L.T. of b.s. of the 2 ODEs, we get

$$sY_1(s) - 2 = 2Y_1(s) + Y_2(s) \text{ and } sY_2(s) = 4Y_1(s) + 2Y_2(s) + 64e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) \quad 1$$

i.e.

$$(s-2)Y_1 - Y_2 = 2 \text{ and } 4Y_1 - (s-2)Y_2 = -64e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$\hookrightarrow Y_2 = (s-2)Y_1 - 2 \longrightarrow 4Y_1 - (s-2)[(s-2)Y_1 - 2] = -64e^{-s}\left(\frac{s+1}{s^2}\right)$

$$(-s^2 + 4s)Y_1 = -2(s-2) - 64e^{-s}\left(\frac{s+1}{s^2}\right)$$

$$Y_1 = \frac{2(s-2)}{s(s-4)} + 64e^{-s}\left(\frac{s+1}{s^3(s-4)}\right) \quad * \quad 2$$

Now $\frac{s-2}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4} \Rightarrow s-2 = A(s-4) + Bs$

Then $s = 0 \Rightarrow A = \frac{1}{2}, \quad s = 4 \Rightarrow B = \frac{1}{2}$

Also $\frac{s+1}{s^3(s-4)} = \frac{C}{s} + \frac{D}{s^2} + \frac{E}{s^3} + \frac{F}{s-4}$

$\Rightarrow s+1 = Cs^2(s-4) + Ds(s-4) + E(s-4) + Fs^3$

$s = 0 \Rightarrow 1 = -4E \Rightarrow E = -1/4$

$s = 4 \Rightarrow 5 = 64F \Rightarrow F = 5/64$

Coeff of s^3: $0 = C + F \Rightarrow C = -5/64.$

Coeff of s^2: $0 = -4C + D \Rightarrow D = -5/16$

Coeff of s: $1 = -4D + E \Rightarrow E = -1/4$ as before.

So now $Y_1(s) = \frac{1}{s} + \frac{1}{s-4} + 64e^{-s}\left[\frac{-5/64}{s} - \frac{5/16}{s^2} - \frac{1/4}{s^3} + \frac{5/64}{s-4}\right] \quad 2$

Since $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$, $\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2}t^2$ & $\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) = e^{4t}$

we get

$$y_1(t) = 1 + e^{4t} + 64\left[-\frac{5}{64} - \frac{5}{16}(t-1) - \frac{1}{8}(t-1)^2 + \frac{5}{64}e^{4(t-1)}\right]u(t-1)$$

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i.e. $y_1(t) = 1 + e^{4t} - [5 + 20(t-1) + 8(t-1)^2 - 5e^{4(t-1)}]u(t-1)$

$= 1 + e^{4t} - [-7 + 4t + 8t^2 - 5e^{4(t-1)}]u(t-1)$ 2

Then $Y_2 = (s-2)Y_1 - 2 = \frac{s-2}{s} + \frac{s-2}{s-4} + 2 + 64e^{-s}[$

$= \frac{2(s-2)^2}{s(s-4)} - 2 + 64e^{-s}\left[\frac{(s+1)(s-2)}{s^3(s-4)}\right]$

$= \frac{2(s^2-4s+4) - 2s^2 + 8s}{s(s-4)} + 64e^{-s}\left[\frac{s^2-s-2}{s^3(s-4)}\right]$

$= \frac{8}{s(s-4)} + 64e^{-s}\left[\frac{s^2-s-2}{s^3(s-4)}\right]$

$= \left[\frac{A'}{s} + \frac{B'}{s-4}\right] + 64e^{-s}\left[\frac{C'}{s} + \frac{D'}{s^2} + \frac{E'}{s^3} + \frac{F'}{s-4}\right]$

Proceed as before, to get $A' = -2$, $B' = 2$ and also

$C' = \frac{-5}{32}$, $D' = \frac{3}{8}$, $E' = \frac{1}{2}$, $F' = \frac{5}{32}$

So $Y_2(s) = -\frac{2}{s} + \frac{2}{s-4} + e^{-s}\left[-\frac{10}{s} + \frac{24}{s^2} + \frac{32}{s^3} + \frac{10}{s-4}\right]$ 2

& then

$y_2(t) = -2 + 2e^{4t} + [-10 + 24(t-1) + 16(t-1)^2 + 10e^{4(t-1)}]u(t-1)$

$= -2 + 2e^{4t} + [16t^2 - 8t - 18 + 10e^{4(t-1)}]u(t-1)$ 2

Meaning: For $0 < t \leq 1$ the solution is

$y_1(t) = 1 + e^{4t}$

$y_2(t) = -2 + 2e^{4t}$

$\left(\Rightarrow \begin{array}{l} y_1(1) = 1 + e^4 \\ y_2(1) = -2 + 2e^4 \end{array}\right)$

[satisfying $y_1' = 2y_1 + y_2$, $y_2' = 4y_1 + 2y_2$

& $y_1(0) = 2$, $y_2(0) = 0$.]

and for $t \geq 1$, the solution is

$y_1(t) = 1 + e^{4t} - [-7 + 4t + 8t^2 - 5e^{4(t-1)}]$

$y_2(t) = -2 + 2e^{4t} + [16t^2 - 8t - 18 + 10e^{4(t-1)}]$

{ satisfying $y_1' = 2y_1 + y_2$, $y_2' = 4y_1 + 2y_2 + 64t$

and $y_1(1) = 1 + e^4$, $y_2(1) = -2 + 2e^4$, so

overall sol^n is conts. at $t = 1$.]