MATH2100 Solutions 6 (= MATH2011 Solutions 1) AJB

1. $$\int x^2\cos(\alpha x)\,dx = x^2\frac{\sin(\alpha x)}{\alpha} - \int 2x\,\frac{\sin(\alpha x)}{\alpha}\,dx$$

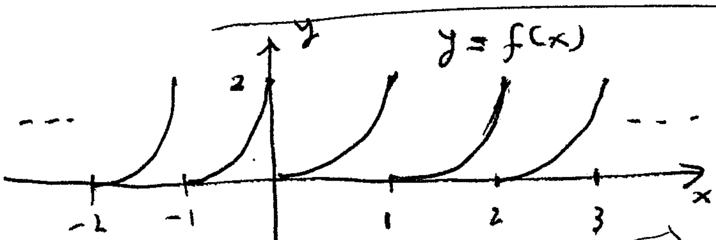
$$= x^2\frac{\sin(\alpha x)}{\alpha} + 2x\,\frac{\cos(\alpha x)}{\alpha^2} - \int 2\,\frac{\cos(\alpha x)}{\alpha^2}\,dx$$

$$= x^2\frac{\sin(\alpha x)}{\alpha} + \frac{2x\cos(\alpha x)}{\alpha^2} - \frac{2\sin(\alpha x)}{\alpha^3} + c$$

$$\int x^2\sin(\alpha x)\,dx = -x^2\frac{\cos(\alpha x)}{\alpha} + \int 2x\,\frac{\cos(\alpha x)}{\alpha}\,dx$$

$$= -x^2\frac{\cos(\alpha x)}{\alpha} + \frac{2x\sin(\alpha x)}{\alpha^2} - \int \frac{2\sin(\alpha x)}{\alpha^2}\,dx$$

$$= -x^2\frac{\cos(\alpha x)}{\alpha} + \frac{2x\sin(\alpha x)}{\alpha^2} + \frac{2\cos(\alpha x)}{\alpha^3} + c$$



$p = 1 = 2L,\ L = \frac{1}{2}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(2n\pi x) + \sum_{n=1}^{\infty} b_n\sin(2n\pi x)$$

$$a_0 = \frac{1}{2L}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx = \int_0^1 f(x)\,dx$$

$$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}.$$

$$a_n = \frac{1}{L}\int_0^1 f(x)\cos(2n\pi x)\,dx = 4\int_0^1 x^2\cos(2n\pi x)\,dx$$

$$= 4\left[\underbrace{\cancel{x^2\frac{\sin(2n\pi x)}{2n\pi}}}_{0} + \frac{2x\cos(2n\pi x)}{(2n\pi)^2} - \underbrace{\cancel{\frac{2\sin(2n\pi x)}{(2n\pi)^3}}}_{0}\right]_0^1$$

$$= \frac{8}{(2n\pi)^2} = \frac{2}{(n\pi)^2}$$

$$b_n = \frac{1}{L}\int_0^1 f(x)\sin(2n\pi x)\,dx = 4\int_0^1 x^2\sin(2n\pi x)\,dx$$

$$= 4\left[-x^2\frac{\cos(2n\pi x)}{2n\pi} + \underbrace{\cancel{\frac{2x\sin(2n\pi x)}{(2n\pi)^2}}}_{0} + \frac{2\cos(2n\pi x)}{(2n\pi)^3}\right]_0^1$$

$$\frac{2}{(2n\pi)^3}(1-1) = 0$$

$$= 4\left(\frac{-1}{2n\pi}\right) = \frac{-2}{n\pi}$$

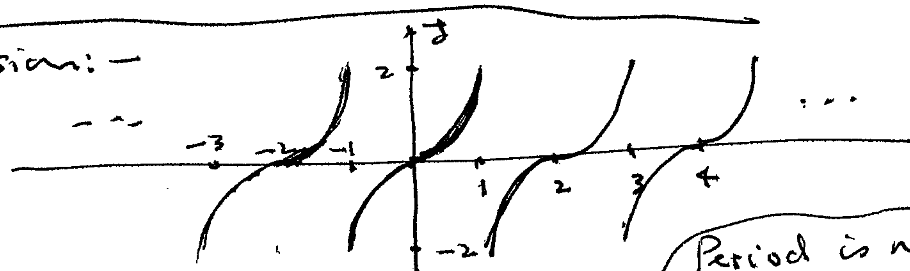
So $F.S. = \frac{2}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(2n\pi x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\pi x)$

We know ~~that~~ this converges to $f(x)$, except at $x = 0, \pm 1, \pm 2, \dots$ where it converges to the value $\frac{1}{2}(0+2) = 1$. So at $x=0$ we have

$$1 = \frac{2}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot 1 - 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Odd periodic extension:–



Period is now 2
$p=2 \quad L=1$

$$F.S. = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$b_n = \frac{2}{L}\int_0^1 2x^2 \sin(n\pi x)\,dx$$

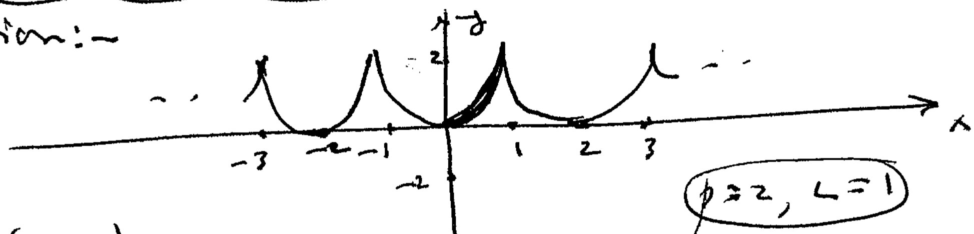
$$= 4\left[-x^2\frac{\cos(n\pi x)}{n\pi} + \cancel{2x\frac{\sin(n\pi x)}{(n\pi)^2}} + \frac{2\cos(n\pi x)}{(n\pi)^3}\right]_0^1$$

$$= 4\left[-\frac{\cos(n\pi)}{n\pi} + \frac{2}{(n\pi)^3}\left[\cos(n\pi) - 1\right]\right] = 4\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}\begin{cases} 0 & n \text{ even} \\ -2 & n \text{ odd}\end{cases}\right]$$

So Fourier sine series is $4\left\{\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right)\sin(\pi x) - \frac{1}{2\pi}\sin(2\pi x) + \left(\frac{1}{3\pi} - \frac{4}{27\pi^3}\right)\sin(3\pi x) \dots\right\}$

This will converge to $F(x)$ for $0 < x < 1$, but will equal 0 at $x=1$

Even periodic extension:–



$p=2, \; L=1$

$$F.S. = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_0 = \frac{1}{L}\int_0^1 F(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}.$$

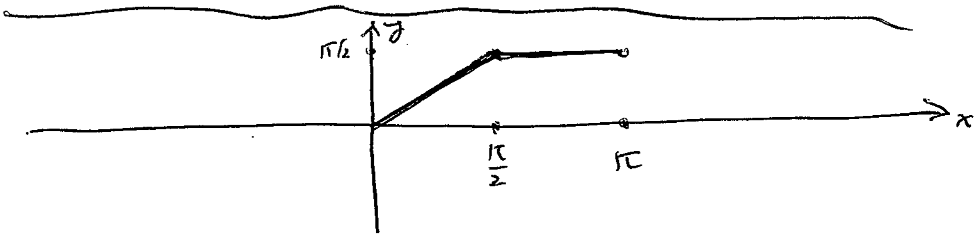
$$a_n = \frac{2}{L}\int_0^1 \underbrace{F(x)}_{2x^2}\cos(n\pi x)\,dx = 4\left[x^2\frac{\sin(n\pi x)}{n\pi} + 2x\frac{\cos(n\pi x)}{(n\pi)^2} - \frac{2\sin(n\pi x)}{(n\pi)^3}\right]_0^1$$

$$= \frac{8\cos(-n\pi)}{(n\pi)^2} = \frac{8}{\pi^2}\frac{(-1)^n}{n^2}$$

So $\quad F.S. = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$

This will equal $F(x)$ for all $-1 \le x < 0$ as even periodic extension is conts.

K11.3 # 22



(a) Half-range cosine series, $\quad L = \pi.$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[\tfrac{\pi}{2}x\right]_{\pi/2}^{\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{\pi^2}{4}\right] = \frac{3\pi}{8}$$

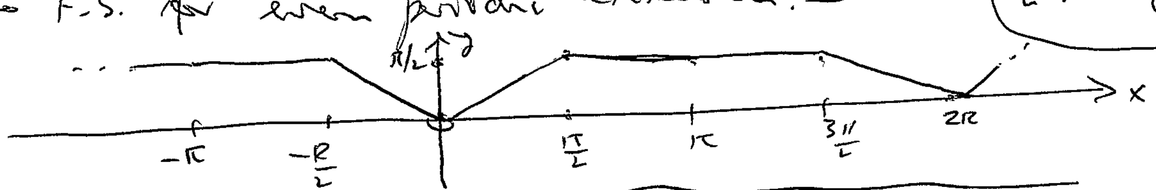
$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi/2} + \frac{\pi}{2n}\left[\sin(nx)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\cancel{\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)} + \frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \cancel{\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)}\right\}$$

So $\quad f(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left\{\sum_{m=0}^{\infty}\frac{-1}{(2m+1)^2}\cos(2m+1)x + \sum_{m=1}^{\infty}\frac{1}{(2m)^2}\left[(-1)^m - 1\right]\cos(2mx)\right\}$

RHS is also F.S. for even periodic extension:— $\quad$ [cf. Qu 20 (b)]



(b) Half-range sine series. $\quad f(x) = \sum_{n=1}^{\infty} b_n\sin(nx)$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\,dx + \frac{2}{\pi}\cdot\frac{\pi}{2}\int_{\pi/2}^{\pi}\sin(nx)\,dx$$

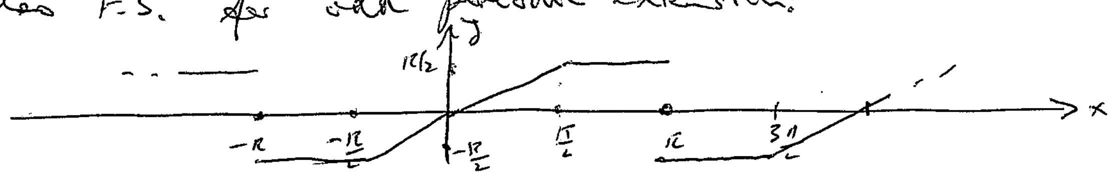
$$= \frac{2}{\pi}\left\{\left[-x\frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi/2} - \frac{\pi}{2n}\left[\cos(nx)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\cancel{-\frac{\pi}{2}\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{n}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} - \frac{\pi}{2n}\cos(n\pi) + \cancel{\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)}\right\}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2m \\ (-1)^{m+1}, & n = 2m+1 \end{cases} \qquad \cos(n\pi) = \begin{cases} 1 & n = 2m \\ -1 & n = 2m+1 \end{cases}$$

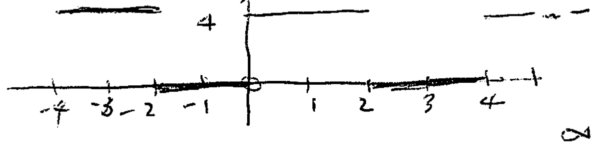
So $\quad f(x) = \frac{2}{\pi}\left\{\sum_{m=0}^{\infty}\left[\frac{(-1)^{m+1}}{(2m+1)^2} + \frac{\pi}{2(2m+1)}\right]\sin(2m+1)x - \frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{2m}\sin(2mx)\right\}$

RHS is also F.S. for odd periodic extension:



Note series converges to 0 at $x = \pi$.

K 11.2 # 2



$p = 2L = 4 \qquad L = 2$

6.X

$$F.S. = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \int_0^2 dx = \qquad a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= 2\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = 2\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx = -\left(\frac{4}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}\left[(-1)^n - 1\right]$$

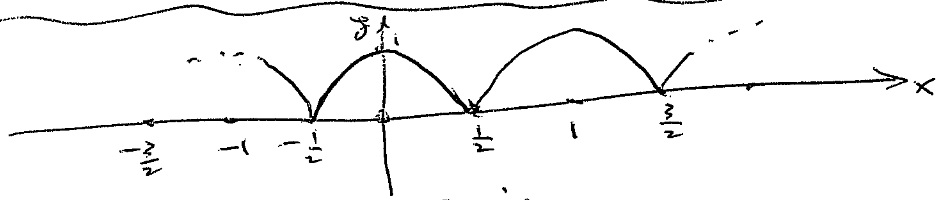
$$= \begin{cases} 0 & n \text{ even} \\ \frac{8}{n\pi} & n \text{ odd} \end{cases}$$

$$\therefore F.S. = 2 + \frac{8}{\pi}\left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)\cdots\right)$$

$$= 2 + \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\sin\frac{(2m+1)\pi x}{2}$$

$\boxed{\text{See } f(x) = F.S. \text{ except at } x = 0, \pm 2, \pm 4, \text{ unless define } f(x) = 2 \text{ there}}$

K11.2 #6



$p = 1 \qquad L = \frac{1}{2}$

Even fn. — cosine series

$$F.S. = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi x)$$

$$a_0 = 1\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi x)\,dx = \frac{1}{\pi}\left[\sin(\pi x)\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2}{\pi}.$$

$$a_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi x)\cos(2n\pi x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\cos(2n+1)\pi x + \cos(2n-1)\pi x\right]dx$$

$$= \left[\frac{1}{2n+1}\sin(2n+1)\pi x + \frac{1}{2n-1}\sin(2n-1)\pi x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{2}{2n+1}\sin\frac{(2n+1)\pi}{2} + \frac{2}{2n-1}\sin(2n-1)\frac{\pi}{2}.$$

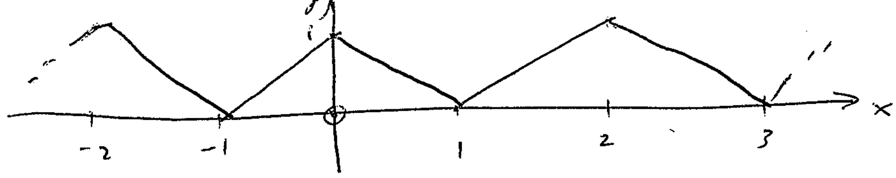
$$= \frac{2}{2n+1}(-1)^n + \frac{2}{2n-1}(-1)^{n+1} = (-1)^n \cdot 2\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= -4(-1)^n\frac{1}{(2n+1)(2n-1)}$$

See in this case F.S. = $f(x)$ everywhere

$$f(x) = \frac{2}{\pi} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}\cos(2n\pi x).$$

K 11.2 #8



$p=2$
$L=1$

(6.5)

Even fn — cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

— series will converge everywhere to the conts. function whose graph is shown.

$$a_0 = \frac{1}{L}\int_0^1 f(x)\,dx = \int_0^1 (1-x)\,dx = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

$$a_n = \frac{2}{L}\int_0^1 (1-x)\cos(n\pi x)\,dx = 2\left[\frac{\cancel{\sin(n\pi x)}}{n\pi}\right]_0^1 - 2\int_0^1 x\cos(n\pi x)\,dx.$$

$$\int x\cos(n\pi x)\,dx = \frac{x\sin(n\pi x)}{n\pi} + \frac{1}{(n\pi)^2}\cos(n\pi x) + c$$

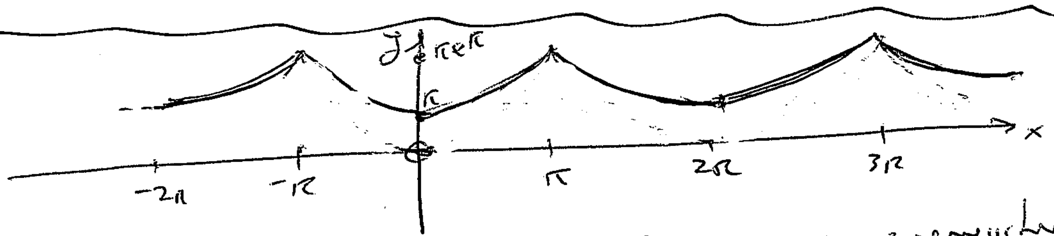
So
$$a_n = -2\left[\frac{\cancel{x\sin n\pi x}}{n\pi} + \frac{1}{(n\pi)^2}\cos n\pi x\right]_0^1$$

$$= \frac{-2}{(n\pi)^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{4}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

So
$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \cdots\right]$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\pi x.$$

K 11.3 #14



$p = 2\pi$
$L = \pi$

Even fn — cosine series — will converge everywhere to conts. ~~function~~ whose graph is shown. $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \pi e^x\,dx = \left[e^x\right]_0^{\pi} = (e^{\pi} - 1).$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \pi e^x\cos(nx)\,dx$$

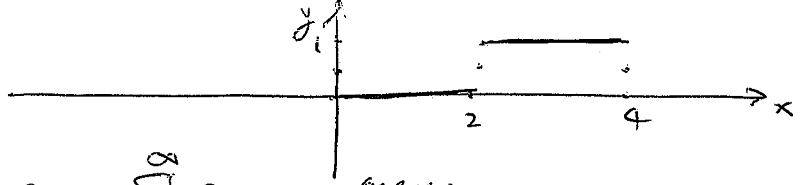
$$\int e^x\cos(\alpha x)\,dx = e^x\frac{\sin(\alpha x)}{\alpha} - \int e^x\frac{\sin(\alpha x)}{\alpha}\,dx = e^x\frac{\sin(\alpha x)}{\alpha} + e^x\frac{\cos(\alpha x)}{\alpha^2} - \int e^x\frac{\cos(\alpha x)}{\alpha^2}\,dx$$

So
$$\left(1 + \frac{1}{\alpha^2}\right)\int e^x\cos(\alpha x)\,dx = e^x\frac{\sin(\alpha x)}{\alpha} + e^x\frac{\cos(\alpha x)}{\alpha^2}$$

So
$$a_n = 2\cdot\left(\frac{n^2}{n^2+1}\right)\left[\frac{\cancel{e^x\sin nx}}{n} + e^x\frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{2n^2}{n^2+1}\left(\frac{1}{n^2}\right)\cdot\left[e^{\pi}(-1)^n - 1\right]$$

So
$$f(x) = (e^{\pi} - 1) + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)\left[e^{\pi}(-1)^n - 1\right]\cos(nx).$$

K11.3 # 20 (6.6)



$(L=4)$

(a) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \qquad 0<x<4$

where $a_0 = \frac{1}{4}\int_0^4 f(x)\,dx = \frac{1}{4}\int_2^4 1\,dx = \frac{1}{2}$

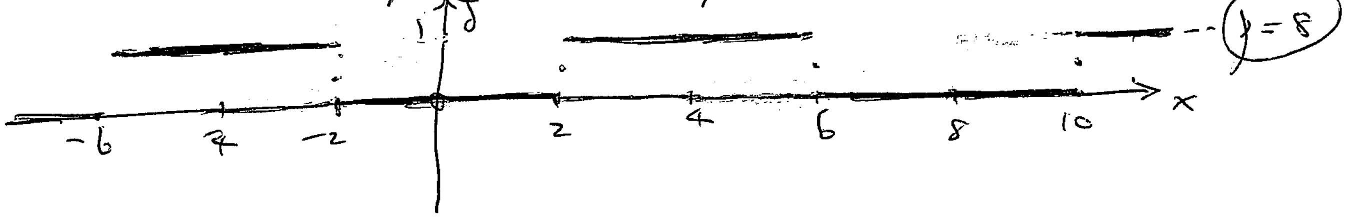
$$a_n = \frac{1}{2}\int_2^4 \cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\cdot\frac{4}{n\pi}\left[\sin\left(\frac{n\pi x}{4}\right)\right]_2^4 = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

So $f(x) = \frac{1}{2} - \frac{2}{\pi}\left[1\cdot\cos\left(\frac{\pi x}{4}\right) + 0\cdot\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cdot\cos\left(\frac{3\pi x}{4}\right) + 0 \cdots\right.$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\cos\frac{(2m+1)\pi x}{4}$$

This will converge to the value $\frac{1}{2}$ at $x = 2$

RHS is also the F.S. for the even periodic extension:—



(b) $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) \qquad 0<x<4$

$$b_n = \frac{1}{2}\int_2^4 \sin\left(\frac{n\pi x}{4}\right)dx = -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi x}{4}\right)\right]_2^4$$

$$= -\frac{2}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$b_1 = -\frac{2}{\pi}[-1-0]$, $b_2 = -\frac{2}{2\pi}[1+1]$, $b_3 = -\frac{2}{3\pi}[-1-0]$, $b_4 = -\frac{2}{4\pi}[1-1]$

$b_5 = -\frac{2}{5\pi}[-1-0]$

$b_6 = -\frac{2}{6\pi}[1+1]$

So $f(x) = +\frac{2}{\pi}\left[\sin\left(\frac{\pi x}{4}\right) \cancel{+} \sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{4}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{4}\right) - \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) \cdots\right]$

$$= +\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\frac{(2m+1)\pi x}{4}$$

$$\cancel{=}\ \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m}\left[1-(-1)^m\right]\sin\left(\frac{m\pi x}{2}\right)$$

using $\cos(n\pi) = (-1)^n$ and

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases}(-1)^m & n=2m\\ 0 & n=2m+1\end{cases}$$

RHS is also the Fourier sine series for the odd periodic extension

Note Values F.S. converges to at $x = 2, 4$