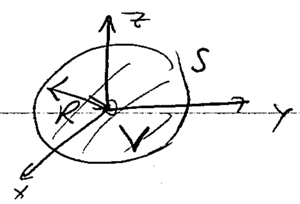
MATH 2100 Solutions 7 (= MATH 2011 Solutions 2) AOB

$$Q(t) = \iiint_V \rho \, dV = \rho_0 e^{-\alpha t} \iiint_V dV = \rho_0 e^{-\alpha t} \tfrac{4}{3}\pi R^3$$



$$\frac{\partial \rho}{\partial t} = -\alpha \rho_0 e^{-\alpha t}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \Rightarrow \underline{\nabla} \cdot \underline{J} = \alpha \rho_0 e^{-\alpha t}$$

If $\underline{J} = f(r,t)\underline{r}, \quad = f(r,t)(x\underline{i} + y\underline{j} + z\underline{k})$

then $\underline{\nabla} \cdot \underline{J} = \frac{\partial}{\partial x}(f x) + \frac{\partial}{\partial y}(f y) + \frac{\partial}{\partial z}(f z)$

$$= 3f + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial}{\partial x}[x^2+y^2+z^2]^{\frac{1}{2}} = \frac{\partial f}{\partial r}\frac{x}{[x^2+y^2+z^2]^{\frac{1}{2}}} = \frac{\partial f}{\partial r}\frac{x}{r}$$

So $\underline{\nabla} \cdot \underline{J} = 3f + \frac{\partial f}{\partial r}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] = 3f + r\frac{\partial f}{\partial r}$

Hence

$\underline{\nabla} \cdot \underline{J} = \alpha \rho_0 e^{-\alpha t} \Rightarrow \quad r\frac{\partial f}{\partial r} + 3f = \alpha \rho_0 e^{-\alpha t}$. 1st order separable ODE

I. Factor r^2 $\Rightarrow \quad r^3\frac{\partial f}{\partial r} + 3r^2 f = \alpha \rho_0 r^2 e^{-\alpha t}$

$$\frac{\partial}{\partial r}[r^3 f] = \alpha \rho_0 r^2 e^{-\alpha t}$$

$$r^3 f = \frac{\alpha \rho_0 r^3}{3} e^{-\alpha t} + B(t)$$

arb. function of t [Careful!]

So $f = \frac{\alpha \rho_0}{3} e^{-\alpha t} + \frac{B(t)}{r^3}$.

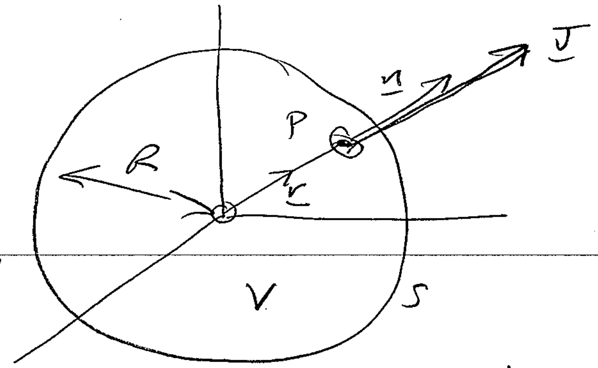
and $\underline{J} = \frac{\alpha \rho_0}{3} e^{-\alpha t}\underline{r} + \frac{B(t)}{r^3}\underline{r}$

$\underline{J} = \underline{0}$ at $\underline{r} = \underline{0} \Rightarrow B(t) = 0$.

So $\underline{J}(\underline{r}, t) = \frac{\alpha \rho_0}{3} e^{-\alpha t}\underline{r}$.

7.2

In this case $\underline{n}$ normal to surface at each point P is in radial direction, as is $\underline{J}$ at that point.



So $\underline{n} = \hat{\underline{r}} = \dfrac{\underline{r}}{r}$, and $\underline{J}\cdot\underline{n} = \frac{1}{3}\alpha\rho_0 e^{-\alpha t}\dfrac{\underline{r}\cdot\underline{r}}{r} = \frac{1}{3}\alpha\rho_0 e^{-\alpha t} r$

$= \frac{1}{3}\alpha\rho_0 e^{-\alpha t} R$ at each pt. on surface

Then $$\oiint_S \underline{J}\cdot\underline{n}\, dS = \frac{1}{3}\alpha\rho_0 e^{-\alpha t} R \underbrace{\oiint_S dS}_{4\pi R^2}$$

$$= \frac{4}{3}\pi R^3 \alpha\rho_0 e^{-\alpha t}$$

$Q(t) = \frac{4}{3}\pi R^3 \rho_0 e^{-\alpha t}$, so $-\dfrac{dQ(t)}{dt} = \frac{4}{3}\pi R^3 \alpha\rho_0 e^{-\alpha t}$

$=$ outflux of Helium over surface.

$$\underline{J} = \underline{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\underline{i} + \frac{\partial\Phi}{\partial y}\underline{j} + \frac{\partial\Phi}{\partial z}\underline{k}$$

So want $\dfrac{\partial\Phi}{\partial x} = \dfrac{\alpha\rho_0}{3}e^{-\alpha t}x$, $\dfrac{\partial\Phi}{\partial y} = \dfrac{\alpha\rho_0}{3}e^{-\alpha t}y$,

$\dfrac{\partial\Phi}{\partial z} = \dfrac{\alpha\rho_0}{3}e^{-\alpha t}z$

$$\Rightarrow \Phi = \frac{\alpha\rho_0}{3}e^{-\alpha t}\left[\tfrac{1}{2}(x^2+y^2+z^2)\right] + \text{const.}$$

Since curl(grad Φ) $= \underline{0}$ for any Φ, it follows ~~that~~ curl $\underline{J} = \underline{0}$

OR Directly
$$\underline{\nabla}\times\underline{J} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \frac{\alpha\rho_0}{3}e^{-\alpha t} = \underline{0}.$$

K9.7 #16: $T = x/(x^2+y^2)$

$$\operatorname{grad} T = \left[\frac{1}{x^2+y^2} + \frac{(-1)x.(2x)}{(x^2+y^2)^2}\right]\underline{i} + \left[\frac{(-x)(2y)}{(x^2+y^2)^2}\right]\underline{j}$$

$$= \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]\underline{i} + \left[\frac{-2xy}{(x^2+y^2)^2}\right]\underline{j}$$

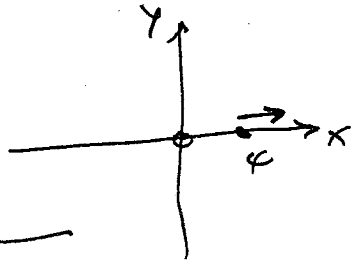
$$= \frac{1}{(x^2+y^2)^2}\left[(y^2-x^2)\underline{i} - 2xy\right]\underline{j}$$

Direction of max. decrease of temp = direction of $(-\operatorname{grad} T)$

Unit vec. in this direction is $\hat{\underline{n}} = -\dfrac{\operatorname{grad} T}{|\operatorname{grad} T|}$

$$= \frac{(x^2-y^2)\underline{i} + 2xy\,\underline{j}}{\left[(x^2-y^2)^2 + 4x^2y^2\right]^{1/2}}$$

$$= \frac{(x^2-y^2)\underline{i} + 2xy\,\underline{j}}{[x^2+y^2]}$$



At $(4,0)$, $\hat{\underline{n}} = \underline{i}$.

K9.7 #18: $T = \sin x \cosh y \Rightarrow \operatorname{grad} T = \cos x \cosh y\,\underline{i} + \sin x \sinh y\,\underline{j}$

Unit vec. in direction of max. decrease of temp. is

$$\hat{\underset{\sim}{n}} = \frac{-(\cos x \cosh y\,\underline{i} + \sin x \sinh y\,\underline{j})}{(\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y)^{1/2}}$$

At $(\tfrac{1}{4}\pi, \ln 5)$, $\cos x = \frac{1}{\sqrt{2}} = \sin x$.

$$\cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{5+\frac{1}{5}}{2} = \frac{13}{5}$$

$$\sinh y = \frac{e^y - e^{-y}}{2} = \frac{5-\frac{1}{5}}{2} = \frac{12}{5}.$$

So
$$\hat{\underset{\sim}{n}} = \frac{-\frac{1}{5\sqrt{2}}(13\underline{i} + 12\underline{j})}{|\ \text{''}\ |} = \frac{13\underline{i} + 12\underline{j}}{\sqrt{13^2+12^2}}$$

$$= \frac{13\underline{i} + 12\underline{j}}{\sqrt{313}}$$

K9.8 #4 $\underline{a} = (x^2+y^2+z^2)^{-3/2}(x\underline{i}+y\underline{j}+z\underline{k}) = \frac{1}{r^3}\underline{r}$

As on p. 7.1 above, $\operatorname{div}\underline{a} = r\frac{d}{dr}(r^{-3}) + \frac{3}{r^3}$.

$= -\frac{3}{r^3} + \frac{3}{r^3} = 0$

So $\operatorname{div}\underline{a} = 0$ (except at $\underline{r}=\underline{0}$, where $\underline{a}$ & its derivatives are undefined.)

K9.8 #8 (a) $\underline{v} = x\underline{i} + y\underline{j} + v_3\underline{k}$

$\operatorname{div}\underline{v} = 1 + 1 + \frac{\partial v_3}{\partial z} = \frac{\partial v_3}{\partial z} + 2$.

Then $\operatorname{div}\underline{v} > 0$ everywhere provided $\frac{\partial v_3}{\partial z} > -2$ everywhere

Eg: $v_3 = 0$, or $v_3 = \alpha z$ $(\alpha > -2)$, or $v_3 = \beta z^3$, or ...

So eg. $\underline{v} = x\underline{i} + y\underline{j}$

(b) Want $\frac{\partial v_3}{\partial z} > -2$ for $-1 < z < 1$

and $\frac{\partial v_3}{\partial z} < -2$ for $|z| > 1$

So, for example, $\frac{\partial v_3}{\partial z} = -2 - (z^2 - 1)$

$= -(1+z^2)$

i.e. $v_3 = c - z - \frac{z^3}{3}$

& $\underline{v} = x\underline{i} + y\underline{j} + (c - z - \frac{z^3}{3})\underline{k}$.

K9.9 #2 $\underline{v} = y^n\underline{i} + z^n\underline{j} + x^n\underline{k}$; $\nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^n & z^n & x^n \end{vmatrix}$

$= \underline{i}(-nz^{n-1}) - \underline{j}(nx^{n-1}) + \underline{k}(-ny^{n-1})$

$= -n[z^{n-1}\underline{i} + x^{n-1}\underline{j} + y^{n-1}\underline{k}]$

K9.9 #6 $\underline{v} = \sin y\,\underline{i} + \cos z\,\underline{j} - \tan x\,\underline{k}$; $\nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \cos z & -\tan x \end{vmatrix}$

$= \underline{i}(\sin z) - \underline{j}(-(1+\tan^2 x)) + \underline{k}(-\cos y)$

$= \sin z\,\underline{i} + (1+\tan^2 x)\,\underline{j} - \cos y\,\underline{j}$