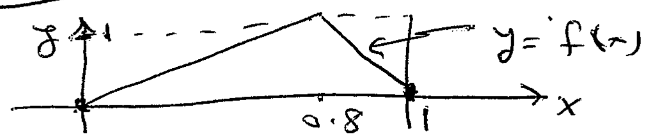
**K 12.3 #10** $\quad u_{tt} = u_{xx} \qquad 0 < x < 1,\ t > 0$




BC $\quad u(0,t) = u(1,t) = 0$

ICs $\quad u(x,0) = f(x) = \begin{cases} \left(\frac{1}{0.8}\right)x = \frac{5}{4}x & 0 < x < 0.8 \\ 5(1-x) & 0.8 < x < 1 \end{cases}$

$u_t(x,0) = g(x) = 0.$

From lectures (pp 27-6, 27-7), solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t)\right] \sin(n\pi x)$$

where $\quad A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx, \quad B_n = \frac{2}{n\pi}\int_0^1 \cancel{g(x)}^{\,0} \sin(n\pi x)\,dx = 0$

$$= (2)\left(\tfrac{5}{4}\right)\int_0^{0.8} x\sin(n\pi x)\,dx + (2)(5)\int_{0.8}^1 (1-x)\sin(n\pi x)\,dx$$

$$= \frac{5}{2}\left\{\left[-\frac{x}{n\pi}\cos(n\pi x)\right]_0^{0.8} + \frac{1}{n\pi}\int_0^{0.8}\cos(n\pi x)\,dx\right\}$$

$$+10\left\{\left[-\frac{(1-x)}{n\pi}\cos(n\pi x)\right]_{0.8}^1 - \frac{1}{n\pi}\int_{0.8}^1 \cos(n\pi x)\,dx\right\}$$

$$= \frac{5}{2}\left\{-\frac{4}{5\pi n}\cos\left(\frac{4n\pi}{5}\right) + \frac{1}{n^2\pi^2}\left[\sin\frac{4n\pi}{5} - \cancel{\sin 0}\right]\right\}$$

$$+10\left\{\frac{1}{5n\pi}\cos\left(\frac{4n\pi}{5}\right) - \frac{1}{n^2\pi^2}\left[\cancel{\sin(n\pi)}^{\,0} - \sin\left(\frac{4n\pi}{5}\right)\right]\right\}$$

$$= \frac{25}{2n^2\pi^2}\sin\left(\frac{4n\pi}{5}\right)$$

Then $\quad u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n\pi t)\sin(n\pi x)$

$$= \frac{25}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{4n\pi}{5}\right)\cos(n\pi t)\sin(n\pi x)$$

Can't simplify further

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$$\eta(x,t) = \frac{x^2}{4c^2t} \Rightarrow \eta_t(x,t) = -\frac{x^2}{4c^2t^2} = -\frac{\eta(x,t)}{t}. \quad (1)$$

$$\Rightarrow \eta_x(x,t) = \frac{2x}{4c^2t} \Rightarrow [\eta_x(x,t)]^2 = \frac{x^2}{4c^4t^2} = \frac{\eta(x,t)}{c^2t}$$

$$\Downarrow$$

$$\eta_{xx}(x,t) = \frac{1}{2c^2t} \quad (1)$$

Now if $u(x,t) = t^{-\frac{1}{2}} f(\eta(x,t))$ and $u_t(x,t) = c^2 u_{xx}(x,t)$

then $u_t(x,t) = -\frac{1}{2} t^{-3/2} f(\eta) + t^{-\frac{1}{2}} \frac{df(\eta)}{d\eta}\,\eta_t$

$$= t^{-\frac{3}{2}}\left[-\tfrac{1}{2} f(\eta) - \eta f'(\eta)\right] \quad (2)$$

and $u_x(x,t) = t^{-\frac{1}{2}} \frac{df(\eta)}{d\eta}\,\eta_x = t^{-\frac{1}{2}} f'(\eta)\,\eta_x$

$$\Rightarrow u_{xx}(x,t)\ t^{-\frac{1}{2}}[f''(\eta)\eta_x]\eta_x + t^{-\frac{1}{2}} f'(\eta)\eta_{xx}$$

$$= \frac{t^{-\frac{3}{2}}}{c^2}\left[f''(\eta)\eta + \tfrac{1}{2} f'(\eta)\right] \quad (2)$$

So $u_t(x,t) = c^2 u_{xx}(x,t)$

$$\Rightarrow t^{-\frac{3}{2}}\left[-\tfrac{1}{2} f(\eta) - \eta f'(\eta)\right] = t^{-\frac{3}{2}}\left[\eta f''(\eta) + \tfrac{1}{2} f'(\eta)\right]$$

$$\Rightarrow \underline{\eta f''(\eta) + (\eta + \tfrac{1}{2}) f'(\eta) + \tfrac{1}{2} f(\eta) = 0} \quad (2)$$

If $f(\eta) = e^{-\eta}$, then $f'(\eta) = -e^{-\eta}$, & $f''(\eta) = e^{-\eta}$

So $\eta f''(\eta) + (\eta + \frac{1}{2}) f'(\eta) + \frac{1}{2} f(\eta)$

$$= \underline{[\eta - (\eta + \tfrac{1}{2}) + \tfrac{1}{2}]e^{-\eta} = 0}. \quad (2)$$

Therefore $u(x,t) = t^{-\frac{1}{2}} e^{-\eta(x,t)}$

$$= t^{-\frac{1}{2}} e^{-x^2/(4c^2t)}$$

is a solution of

$u_t = c^2 u_{xx}$. (1)

K12.3 #2 $u_{tt}(x,t) = u_{xx}(x,t)$ $\quad 0 < x < 1, \quad t > 0$ (8.3)

$u(x,0) = k\left(\sin \pi x - \frac{1}{3}\sin(3\pi x)\right) = f(x), \quad u_t(x,0) = g(x) = 0$

From lecture, sol$^n$ is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t)\right] \sin(n\pi x)$$

$\Rightarrow \quad u(x,0) = \left[A_1 \sin(\pi x) + A_2 \sin(2\pi x) + \dots\right]$

Obviously, $A_1 = k, \quad A_3 = -\frac{1}{3}k$, all other $A_n = 0$.

Also, $B_n = \frac{2}{n\pi}\int_0^1 \cancel{g(x)}^{\,0} \sin(n\pi x)\,dx = 0$.

So $\quad u(x,t) = k\left[\cos(\pi t)\sin(\pi x) - \frac{1}{3}\cos(3\pi t)\sin(3\pi x)\right]$

[ Could check ~~that~~ $A_n$ have these values using

$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$, but you should see

this is unnecessary! ]

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K12.3 #4 As in above example, $B_n = 0$, and

$$A_n = 2k\int_0^1 (x - x^3)\sin(n\pi x)\,dx \quad 1$$

Integration by parts gives

$$\int x \sin(\alpha x)\,dx = -\frac{x}{\alpha}\cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x) + \text{const.} \quad 1$$

$$\int x^3 \sin(\alpha x)\,dx = -\frac{x^3}{\alpha}\cos(\alpha x) + \frac{3x^2}{\alpha^2}\sin(\alpha x) + \frac{6x}{\alpha^3}\cos(\alpha x) - \frac{6}{\alpha^4}\sin(\alpha x) + \text{const} \quad 2$$

and so

$$A_n = 2k\left\{ -\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\cancel{\sin(n\pi x)}^{\,0} + \frac{x^3}{n\pi}\cos(n\pi x) - \frac{3x^2}{n^2\pi^2}\cancel{\sin(n\pi x)}^{\,0} - \frac{6x}{n^3\pi^3}\cos(n\pi x) + \frac{6}{n^4\pi^4}\cancel{\sin(n\pi x)}^{\,0} \right\}_0^1$$

$$= 2k\left\{\left(-\cancel{\frac{1}{n\pi}} + \cancel{\frac{1}{n\pi}} - \frac{6}{n^3\pi^3}\right)\cos(n\pi)\right\}$$

$$= -\frac{12k}{n^3\pi^3}(-1)^n \quad 4$$

So $\quad u(x,t) = \frac{12k}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\cos(n\pi t)\sin(n\pi x) \quad 1$

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