MATH2100 Solutions 9 (= MATH2011 Solutions 4)

1. PDE $u_t(x,t) = c^2 u_{xx}(x,t)$, $0 < x < L$, $t > 0$ (a)

BCs $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$ (b) (c)

IC $u(x,0) = f(x)$ $0 < x < L$ (d)

Look for all solutions of homogeneous equations (a), (b), (c) in the separated form ①

$$u(x,t) = F(x)G(t) \quad -(e)$$

$(b) \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0$ [or $G(t) = 0 \Rightarrow u(x,t) = 0$ <u>trivial</u>] — (f) — ①

(c) $\overset{\text{similarly}}{\Rightarrow}$ $F(L)G(t) = 0 \Rightarrow F(L) = 0$ — (g)

Now (e) in (a) $\Rightarrow$ $F(x)G'(t) = c^2 F''(x)G(t)$

$\div c^2 F(x)G(t)$ $\Rightarrow$ $\dfrac{G'(t)}{c^2 G(t)} = \dfrac{F''(x)}{F(x)} = k$ (separation constant)

$\Rightarrow$ $\underbrace{G'(t) = kc^2 G(t)}_{(m)}$ — ① <u>and</u> — $F''(x) = kF(x)$ ~ (n)

Look at (n) + (f) + (g) :—

<u>Case (a)</u> $k = 0$: $F''(x) = 0 \Rightarrow F(x) = Ax + B$

$(f) \Rightarrow$ $0 = F(0) = 0 + B \Rightarrow F(x) = Ax$

then $(g) \Rightarrow$ $0 = F(L) = AL \Rightarrow A = 0$ ①

$\Rightarrow F(x) = 0 \Rightarrow u(x,t) = 0$ <u>trivial</u>

<u>Case (b)</u> $k > 0$: $(k = \mu^2, \mu > 0)$

$F''(x) = \mu^2 F(x) \Rightarrow F(x) = A\cosh(\mu x) + B\sinh(\mu x)$

$(f) \Rightarrow$ $0 = F(0) = A + 0 \Rightarrow F(x) = B\sinh(\mu x)$

then $(g) \Rightarrow$ $0 = F(L) = B\sinh(\mu L) \Rightarrow B = 0$ ①

$\Rightarrow F(x) = 0 \Rightarrow u(x,t) = 0$ <u>trivial</u>.

<u>Case (c)</u> $k < 0$: $(k = -p^2, p > 0)$

$F''(x) = -p^2 F(x) \Rightarrow F(x) = A\cos(px) + B\sin(px)$

$$(f) \Rightarrow \quad 0 = F(0) = A + 0 \Rightarrow F(x) = B\sin(px)$$

$$(g) \Rightarrow \quad 0 = F(L) = B\sin(pL) \Rightarrow B = 0 \quad (\Rightarrow F(x) = 0 \Rightarrow u(x,t) = 0 \ \underline{\text{trivial}})$$

$$\underline{\underline{\text{or}}} \quad \sin(pL) = 0$$

$$\Rightarrow \quad p = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots$$

So we get $\overline{F(x) = B\sin\left(\frac{n\pi x}{L}\right)}$ corresp. to $p = \frac{n\pi}{L}$. (2)

$\left[\Leftrightarrow k = -\left(\frac{n\pi}{L}\right)^2\right]$

Corresponding equation (m) is

$$G'(t) = -\left(\frac{n\pi c}{L}\right)^2 G(t)$$

$$\Rightarrow \quad G(t) = C e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

Combining, we have $u(x,t) = B.C.\, e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$ (1)

or

$$u_n(x,t) = B_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3 \ldots$$

Following Fourier, assume $\underline{\text{every}}$ solution of (a) + (b) + (c) has form

$$(*) \quad u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right) \quad (1) \qquad 0 < x < L,\ t > 0,$$

for some choice of the B_n.

Then IC (d) $\Rightarrow$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \cdot 1 \cdot \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L$$

Recognize this as Fourier half-range sine series for $f(x)$.

$$\Rightarrow \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (1)$$

If $f(x) = u_0$,

$$B_n = \frac{2u_0}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\left(\frac{2u_0}{L}\right)\left(\frac{L}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2u_0}{n\pi}\left[1 - \cos(n\pi)\right] = \frac{2u_0}{n\pi}\left[1 - (-1)^n\right] = \begin{cases} 0, & n = 2m \text{ even} \\ \frac{4u_0}{(2m+1)\pi}, & n = 2m+1 \text{ (odd)} \end{cases}$$

Sub. in $(*)$ to get

$$u(x,t) = \frac{4u_0}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)} e^{-\left[\frac{(2m+1)\pi c}{L}\right]^2 t}\sin\left[\frac{(2m+1)\pi x}{L}\right]$$

If $f(x) = \frac{u_0}{c^2} x(L-x)$, then

$$B_n = \frac{2}{L} \cdot \frac{u_0}{L^2} \int_0^L \underbrace{x(L-x)}_{u} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{v'} dx$$

$$= \frac{2u_0}{L^3} \cdot \left\{ \cancel{\left[-\frac{L}{n\pi} x(L-x) \cos\left(\frac{n\pi x}{L}\right)\right]_0^L}^{\,0} + \frac{L}{n\pi} \int_0^L \underbrace{(L-2x)}_{u} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{v'} dx \right\}$$

$$= \frac{2u_0}{L^3} \left\{ \frac{L}{n\pi} \cdot \left(\cancel{\left[\frac{L}{n\pi} \sin\frac{n\pi x}{L} \cdot (L-2x)\right]_0^L}^{\,0} - \frac{L}{n\pi} \int_0^L (-2) \sin\left(\frac{n\pi x}{L}\right) dx \right. \right.$$

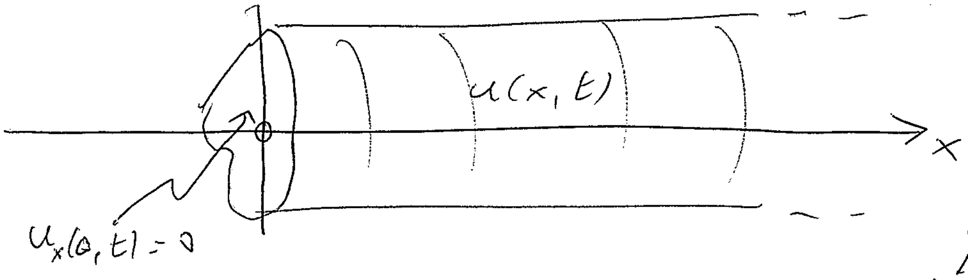
$$= \frac{4u_0}{L n^2 \pi^2} \cdot \left(-\frac{L}{n\pi}\right) \left[\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

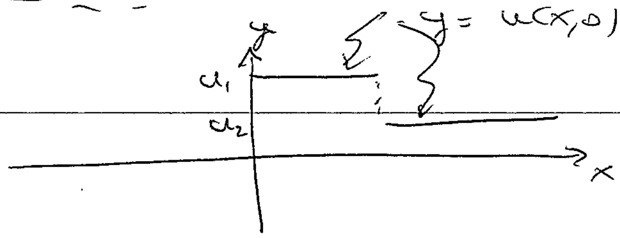
(2) $$= \frac{4u_0}{(n^3\pi^3)} \cdot \left[1 - (-1)^n\right] = \begin{cases} 0 & n = 2m \text{ even} \\ \frac{8u_0}{(2m+1)^3 \pi^3} & n = 2m+1 \text{ odd} \end{cases}$$

$$\therefore \quad u(x,t) = \frac{8u_0}{\pi^3} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} e^{-\left[\frac{(2m+1)\pi c}{L}\right]^2 t} \sin\frac{(2m+1)\pi x}{L}$$

(1) $$= \frac{8u_0}{\pi^3} \left\{ e^{-\frac{\pi^2 c^2 t}{L^2}} \sin\left(\frac{\pi x}{L}\right) + \frac{1}{27} e^{-\frac{9\pi^2 c^2 t}{L^2}} \sin\left(\frac{3\pi x}{L}\right) + \frac{1}{125} e^{-\frac{25\pi^2 c^2 t}{L^2}} \sin\left(\frac{5\pi x}{L}\right) \cdots \right\}$$

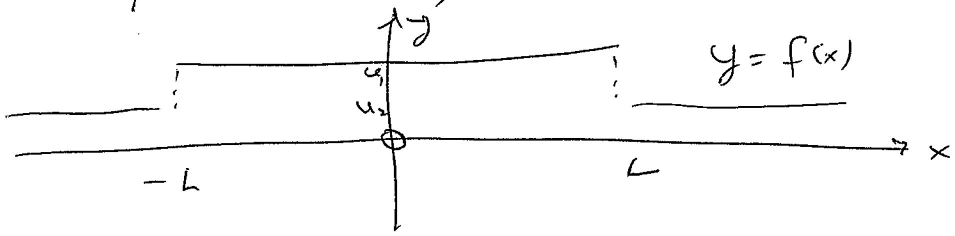
2.





$$u(x,0) = F(x) = \begin{cases} u_1 & 0 < x < L \\ u_2 & x > L \end{cases}$$

Because of BC at $x = 0$, extend to even data :—



(2)

$$f(x) = \begin{cases} u_1 & -L < x < L \\ u_2 & |x| > L. \end{cases}$$

Solve problem on $-\infty < x < \infty$, ignoring BC, which will be satisfied automatically, by symmetry.

Then
$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\,dy \qquad (2)$$

$$= \frac{u_1}{\sqrt{4\pi c^2 t}} \int_{-L}^{L} e^{-(x-y)^2/4c^2t}\,dy + \frac{u_2}{\sqrt{4\pi c^2 t}} \left\{ \int_{-\infty}^{-L} + \int_{L}^{\infty} e^{-(x-y)^2/4c^2t}\,dy \right\}$$

In each integral, put $\frac{x-y}{\sqrt{4c^2t}} = v \iff y = x - \sqrt{4c^2t}\,v$

so $dy = -\sqrt{4c^2t}\,dv$

& $y = \infty \iff v = -\infty \qquad y = -\infty \iff v = +\infty$

$y = L \iff v = \frac{x-L}{\sqrt{4c^2t}} \qquad y = -L \iff v = \frac{x+L}{\sqrt{4c^2t}}$

So
$$u(x,t) = \frac{u_1}{\sqrt{\pi}} \int_{(x+L)/\sqrt{\ }}^{(x-L)/\sqrt{\ }} e^{-v^2}(-dv) + \frac{u_2}{\sqrt{\pi}} \left\{ \int_{\infty}^{(x+L)/\sqrt{\ }} + \int_{(x-L)/\sqrt{\ }}^{-\infty} e^{-v^2}(-dv) \right\} \qquad (2)$$

$$= \frac{u_1}{2}\,\frac{2}{\sqrt{\pi}} \left\{ \int_0^{(x+L)/\sqrt{\ }} - \int_0^{(x-L)/\sqrt{\ }} e^{-v^2}dv \right\} + \frac{u_2}{2}\,\frac{2}{\sqrt{\pi}} \left\{ -\int_0^{(x+L)/\sqrt{\ }} + \underbrace{\int_0^{\infty}}_{\frac{\sqrt{\pi}}{2}} + \underbrace{\int_{-\infty}^{0}}_{\frac{\sqrt{\pi}}{2}} + \int_0^{(x-L)/\sqrt{\ }} e^{-v^2}dv \right\}$$

$$= u_2 + \frac{u_1 - u_2}{2} \left[\operatorname{erf}\left(\frac{x+L}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{4c^2t}}\right) \right] \qquad (1)$$