Q 1 (a). Show that the Fourier Series corresponding to the function defined by

$$f(x) = \alpha\,, \qquad -L < x \leq 0\,; \qquad f(x) = \beta\,, \qquad 0 < x \leq L\,;$$
$$\text{and} \qquad f(x+2L) = f(x)\,, \qquad -\infty < x < \infty\,,$$

where α and β are constants, is given by

$$\frac{1}{2}(\alpha+\beta) - \frac{2(\alpha-\beta)}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin\left[\frac{(2m-1)\pi x}{L}\right]. \qquad (\star)$$

(17 marks)

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \quad 1$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0}\alpha\,dx + \frac{1}{2L}\int_{0}^{L}\beta\,dx = \frac{1}{2}(\alpha+\beta) \quad 3$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{0}\alpha\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L}\beta\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left\{\alpha\,\cancel{\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Big]_{-L}^{0}} + \beta\,\cancel{\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Big]_{0}^{L}}\right\}$$

0 0 0 0

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$$= 0$$

Question 1 continued on next page.

Q 1 (a). **Working space only**

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{0}\alpha \sin\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L}\beta \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left\{ -\frac{\alpha L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big]_{-L}^{0} - \frac{\beta L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big]_{0}^{L}\right\}$$

$$= -\frac{\alpha}{n\pi}\underbrace{\left[1-\cos(-n\pi)\right]}_{=\,1-(-1)^n} - \frac{\beta}{n\pi}\underbrace{\left[\cos(n\pi)-1\right]}_{=\,(-1)^n-1}$$

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$$= -\frac{1}{n\pi}(\alpha-\beta)\left[1-(-1)^n\right]$$

$$b_1 = -\frac{1}{\pi}(\alpha-\beta).2, \quad b_2 = 0, \quad b_3 = -\frac{1}{3\pi}(\alpha-\beta)2, \ldots$$

Then $$f(x) = \frac{1}{2}(\alpha+\beta) - \frac{2(\alpha-\beta)}{\pi}\left[1\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \cdots\right]$$

$$= \frac{1}{2}(\alpha+\beta) - \frac{2(\alpha-\beta)}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin\left[\frac{(2m-1)\pi x}{L}\right].$$

2

Question 1 continued on next page.

Q 1 (b). Briefly explain the value taken by the series ($\star$) at $x = 0$.

(5 marks)

At $x=0$, series converges trivially to $\frac{1}{2}(\alpha+\beta)$ 1

At the jump discontinuity, f has well defined limits from L and R, and so does f'

$$f(0_-) = \alpha, \quad f(0_+) = \beta \qquad 1$$

$$f'(0_-) = 0, \quad f'(0_+) = 0. \qquad 1$$

Then we know series will converge to average value 2 across jump.

Q 1 (c). By considering the value to which the series ($\star$) must converge at $x = L/2$ deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

(6 marks)

We know series will converge to $f(\frac{1}{2}L) = \beta$ as function is cont. & diffble there. 2

So

$$\beta = \frac{1}{2}(\alpha+\beta) - 2\left(\frac{\alpha-\beta}{\pi}\right) \sum_{m=1}^{\infty} \left(\frac{1}{2m-1}\right) \sin\left[\frac{(2m-1)\pi}{2}\right]$$

$$\frac{1}{2}(\beta-\alpha) = 2\left(\frac{\beta-\alpha}{\pi}\right)\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right] \qquad 2$$

Must hold whether or not $\alpha \neq \beta$. So

$$\frac{1}{2} = \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4} \qquad 2$$

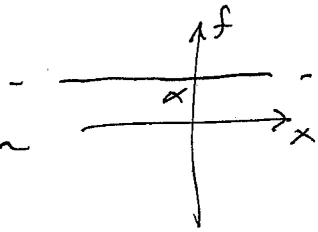
Question 1 continued on next page. TURN OVER

Q 1 (d). Describe and explain in a few words what happens to the series ($\star$) when (a) $\alpha = \beta$ and (b) $\alpha = -\beta$.

(5 marks)

(a) When $\alpha = \beta$.

$f(x) = \alpha$ (const.) — even fn



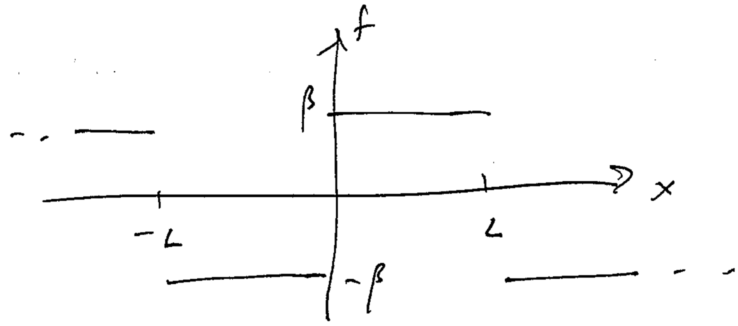
& series collapses to value α.

— even function with ~~[illegible]~~ cosine series

2

(b) When $\alpha = -\beta$

$f(x)$ is odd fn



with sine series:

$$f(x) = \frac{4\beta}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin\left[\frac{(2m-1)\pi x}{L}\right]$$

3

Question 2 see next page.

Q 2 (a). You are given (**no need to check!**) that the function $G(x-y,t)$ defined by

$$G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-(x-y)^2/(4c^2t)}$$

satisfies

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t)\,, \qquad -\infty < x < \infty\,, \qquad t > 0\,.$$

Show that $u(x,t)$ defined by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

satisfies the 1-dimensional Heat Equation for $-\infty < x < \infty$ and $t > 0$, and also the initial condition

$$\lim_{t\to 0+} u(x,t) = f(x)\,, \qquad -\infty < x < \infty\,.$$

(9 marks)

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

$$u_t(x,t) = \int_{-\infty}^{\infty} G_t(x-y,t) f(y)\, dy$$

$$u_{xx}(x,t) = \int_{-\infty}^{\infty} G_{xx}(x-y,t) f(y)\, dy$$

So $$u_t - c^2 u_{xx} = \int_{-\infty}^{\infty} [G_t(x-y,t) - c^2 G_{xx}(x-y,t)]\, f(y)\, dy$$

$$= 0.$$

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Question 2 continued on next page.

Q 2 (a). **Working space only**

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\, dy$$

Put $v = \frac{x-y}{\sqrt{4c^2t}} \Leftrightarrow y = x - \sqrt{4c^2t}\, v$

$dy = -\sqrt{4c^2t}\, dv$

$y = -\infty \Leftrightarrow v = \infty$

$y = +\infty \Leftrightarrow v = -\infty$

So $u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{+\infty}^{-\infty} e^{-v^2} f(x - \sqrt{4c^2t}\, v)(-dv)\sqrt{4c^2t}$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} f(x - \sqrt{4c^2t}\, v)\, dv \quad 3$$

As $t \to 0_+$

$$u(x,t) \Rightarrow \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} f(x)\, dv$$

$$= f(x) \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv}_{=1}$$

3

$$= f(x)$$

Question 2 continued on next page.

Q 2 (b). A very long cylindrical iron bar, with unknown thermal diffusivity c^2, lies along the positive x-axis. The sides of the bar and the end at $x = 0$ are thermally insulated, and the temperature u inside the bar is a function only of x and of time t. The temperature distribution in the bar at $t = 0$ is given by

$$u(x,0) = F(x) = \begin{cases} u_0 \text{ (const.)} & , \quad 0 < x < L \\ 0 & , \quad L < x < \infty \end{cases} .$$

Deduce that

$$u(x,t) = \frac{1}{2}u_0 \left\{ \operatorname{erf}\left(\frac{x+L}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{4c^2t}}\right) \right\}$$

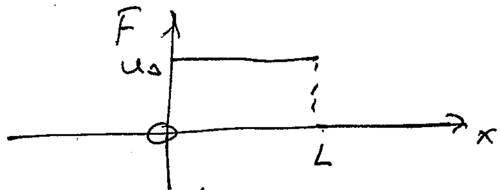
for $x > 0$, $t > 0$, where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv .$$

(18 marks)

We have to solve

$$③ \left\{ \begin{array}{lll} & u_t(x,t) = c^2 u_{xx}(x,t) & x>0,\ t>0 \\ BC & u_x(0,t) = 0 & \\ IC & u(x,0) = F(x) & \end{array} \right.$$

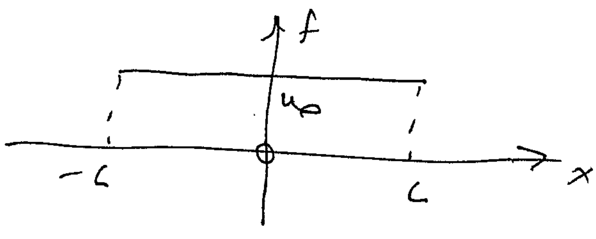


Extend to problem on whole x axis with even initial data, so BC is satisfied automatically:-

So consider $u_t(x,t) = c^2 u_{xx}(x,t) \quad -\infty < x < \infty,\ t > 0$

IC $\quad u(x,0) = f(x)$

③



Sol^n: $$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\, dy \quad ①$$

Question 2 continued on next page.

Q 2 (b). **Working space only**

$$= \frac{u_0}{\sqrt{4\pi c^2 t}} \int_{-L}^{L} e^{-(x-y)^2/4c^2t}\, dy$$

$$v = \frac{x-y}{\sqrt{4c^2t}} \iff y = x - \sqrt{4c^2t}\, v$$
$$dy = -\sqrt{4c^2t}\, dv \quad 3$$

$$y = \pm L \iff v = \frac{x \mp L}{\sqrt{4c^2t}}$$

So

$$u(x,t) = \frac{u_0}{\sqrt{4\pi c^2 t}} \int_{\frac{x+L}{\sqrt{4c^2t}}}^{\frac{x-L}{\sqrt{4c^2t}}} e^{-v^2} \left(-\sqrt{4c^2t}\right) dv \quad 3$$

$$= \frac{u_0}{\sqrt{\pi}} \int_{\frac{x-L}{\sqrt{4c^2t}}}^{\frac{x+L}{\sqrt{4c^2t}}} e^{-v^2}\, dv \quad 2$$

$$= \tfrac{1}{2} u_0 \left\{ \frac{2}{\sqrt{\pi}} \int_0^{\frac{x+L}{\sqrt{4c^2t}}} e^{-v^2}\, dv - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-L}{\sqrt{4c^2t}}} e^{-v^2}\, dv \right\} \quad 3$$

$$= \tfrac{1}{2} u_0 \left\{ \operatorname{erf}\left(\frac{x+L}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{4c^2t}}\right) \right\} \quad 2$$

Question 2 continued on next page.

Q 2 (c). Given that erf(0.5)$\approx$ 0.5, deduce that the thermal diffusivity c^2 is given approximately by

$$c^2 = L^2/t_1 \, ,$$

where t_1 is the time taken for the temperature at the face $x = 0$ to reach the value $u_0/2$.

(5 marks)

At $x=0$

$$u(0,t) = \tfrac{1}{2} u_0 \left\{ \operatorname{erf}\left(\frac{L}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{-L}{\sqrt{4c^2t}}\right)\right\}$$

$$= u_0 \operatorname{erf}\left(\frac{L}{\sqrt{4c^2t}}\right) \quad \text{as } \operatorname{erf}(-z) = -\operatorname{erf}(z)$$

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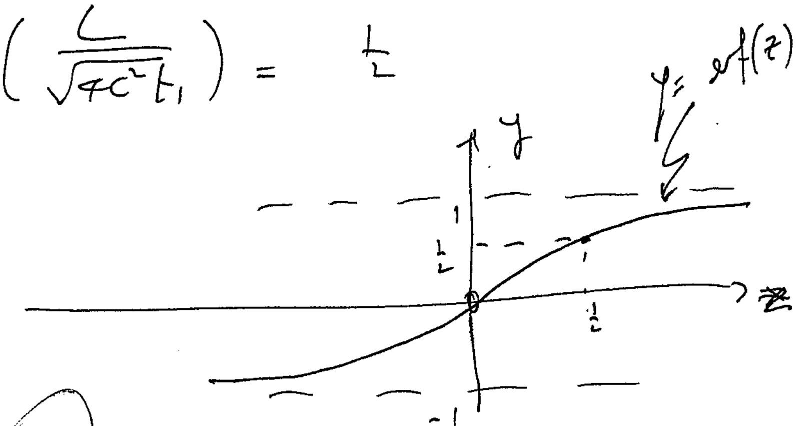
Then

$$u(0,t_1) = \tfrac{1}{2} u_0 = u_0 \operatorname{erf}\left(\frac{L}{\sqrt{4c^2t_1}}\right)$$

$$\Leftrightarrow \operatorname{erf}\left(\frac{L}{\sqrt{4c^2t_1}}\right) = \tfrac{1}{2}$$

$$\Leftrightarrow \frac{L}{\sqrt{4c^2t_1}} \approx \frac{1}{2}$$



$$\Leftrightarrow c^2 \approx \frac{L^2}{t_1}$$

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