APPLICATION - Given 3 points on a circle, find the equation of the circle on which they lie

We can build a theory from some basic knowledge of systems of linear equations, matrices and determinants.

- Know \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) \quad \{3\ text{ equations}\)
  and \((x - a)^2 + (y - b)^2 = r^2\quad \quad (1)\quad \{3\ text{ unknowns} a, b, r\)

- Rearrange (1) to get our unknowns as coefficients \(c_1, c_2, c_3, c_4\) to our \(x's, y's:\)

\[

c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0 \\
c_1(x_1^2 + y_1^2) + c_2x_1 + c_3y_1 + c_4 = 0 \\
c_1(x_2^2 + y_2^2) + c_2x_2 + c_3y_2 + c_4 = 0 \\
c_1(x_3^2 + y_3^2) + c_2x_3 + c_3y_3 + c_4 = 0
\]

\[-(2)\]

\[
\begin{bmatrix}
  x^2 + y^2 & x & y & 1 \\
  x_1^2 + y_1^2 & x_1 & y_1 & 1 \\
  x_2^2 + y_2^2 & x_2 & y_2 & 1 \\
  x_3^2 + y_3^2 & x_3 & y_3 & 1
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]

\(A\quad x\quad 0\)

- Assuming these points really do lie on a circle, there must be a non-trivial solution for \(c_1, c_2, c_3, c_4\) \(\Rightarrow\ A\ must\ be\ singular.\)

  (This is because if \(A^{-1}\) existed, then \(A^{-1}(Ax) = A^{-1}0 \Rightarrow x = 0).\)

- Thus \(det(A) = \)

\[
\begin{vmatrix}
  x^2 + y^2 & x & y & 1 \\
  x_1^2 + y_1^2 & x_1 & y_1 & 1 \\
  x_2^2 + y_2^2 & x_2 & y_2 & 1 \\
  x_3^2 + y_3^2 & x_3 & y_3 & 1
\end{vmatrix} = 0.
\]

Example: Find the equation of the circle with points \((1, 7), (6, 2), (4, 6).\)

This method can be applied to straight lines, parabolas, hyperbolas, ellipses, planes and spheres also.
APPLICATION - Finding the equation of a circle (Part II)

For a linear system $A\mathbf{x} = \mathbf{b}$:

- We can put in an approximate solution $\mathbf{x}_1$ and call
  
  $$r(\mathbf{x}_1) = \mathbf{b} - A\mathbf{x}_1$$

  the residual.

- The approximate solution $\mathbf{x}_1$ where $\|r(\mathbf{x}_1)\|^2$ is minimised is called the **LEAST SQUARES SOLUTION**.

- We don’t need to go into this, but such a solution exists and is unique.

Gauss used the method of least squares to predict the orbit of the asteroid Ceres in 1801.

Manufactured goods such as rods, disks and pipes are circular in shape. Quality control engineers are often employed to test production line standards.
- sensing machines are used to record co-ordinates of points on the perimeter.
- a least squares test is performed to see how close the measured points are to the circle.
APPLICATION - Trends in population migration

Problem: The total population of a large metropolitan area undergoes relatively small variations. Every year, 6% of people living in the city move to the suburbs and 2% of people living in the suburbs move to the city. Suppose initially 30% of the total population lives in the city and 70% of the total population lives in the suburbs. What will the percentages be in 10 years? 30 years?

- Begin by creating an initial state vector, where the first (second) entry corresponds to the percentage of people living in the city (suburbs): \( x_0 = \begin{bmatrix} 0.30 \\ 0.70 \end{bmatrix} \)

- We would like to construct a transformation \( T_A \) that maps the state vector to the state vector for the following year. That is,

\[
T_A(x_0) = Ax_0 = x_1
\]

where

\[
x_1 = \begin{bmatrix} (94\% \text{ of people who initially live in the city} \\
+2\% \text{ of people who initially live in the suburbs}) \\
(98\% \text{ of people who initially live in the suburbs} \\
+6\% \text{ of people who initially live in the city}) \end{bmatrix}
\]

- The following matrix achieves this:

\[
A = \begin{bmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{bmatrix} \quad \text{- entries} \geq 0 \text{ and columns add to 1}
\]

- called a \textit{stochastic matrix}

- In fact, this transformation matrix can be applied to the state vector of any year, \( x_i \), to find the state vector of the following year, \( x_{i+1} \). Thus given the initial state, the state vector for the \( n \)th year is given by \( x_n = A^n x_0 \).

- \( x_{10} = [0.27, 0.73]^t \)  \( x_{30} = [0.25, 0.75]^t \)  \( x_{50} = [0.25, 0.75]^t \)

Note: the sequence of vectors converges to a limit, called a \textit{steady state vector}.

This type of mathematical model is called a \textbf{Markov Process}. 

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Let’s investigate why the system approaches the steady state vector $x_n = [0.25, 0.75]^t$ for large values of $n$.

- For reasons which will become clear, it is useful express $x_0$ in terms of the basis $u_1 = [1, 3]^t$ and $u_2 = [-1, 1]^t$:

$$x_0 = \begin{bmatrix} 0.30 \\ 0.70 \end{bmatrix} = 0.25u_1 - 0.45u_2$$

- After the first year the state vector is given by

$$x_1 = Ax_0 = A(0.25u_1 - 0.45u_2) = 0.25(Au_1) - 0.45(Au_2) = 0.25 \begin{bmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 0.45 \begin{bmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.25u_1 - 0.45(0.92u_2)$$

In other words, $u_1$ and $u_2$ are eigenvectors of $A$, with respective eigenvalues 1 and 0.92.

- It can easily be seen that the general formula has the form

$$x_n = 0.25u_1 - 0.45(0.92)^n u_2$$

This is a useful form, since the second term approaches zero as $n$ becomes large. The first term is indeed the steady state vector that was found earlier.

Thus, finding the eigenvectors and eigenvalues of the transformation matrix $A$ allows us to discover if the sequence of states vectors $x_0, x_1, x_2 \ldots$ converge, and also to calculate such a limit.
APPLICATION - Networks and graphs

The area of mathematics known as graph theory has many useful applications. One application in particular is that of communication networks.

Problem: Suppose there is a communications network involving transmitter/receivers $V_1, V_2, V_3, V_4$ and $V_5$. These points (vertices) can only communicate along the following lines (edges):

```
V1 ——— V4
   |     |
   |     |
   |     |
V5 ——— V3
```

Q1: What is the minimum number of lines needed for $V_1$ to communicate with $V_4$?

Q2: How many different ways are there that $V_3$ can communicate with $V_5$ using exactly 3 lines?

- We construct what is known as an adjacency matrix $A$ to represent the graph. Each $ij$th element of the graph is either a 1 (if vertices $V_i$ and $V_j$ are connected by an edge), or a 0.

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}$$

- Theorem: If $A$ is the adjacency matrix of a graph, then the $ij$th entry of $A^k$ is equal to the number of walks of length $k$ from $V_i$ to $V_j$. (This can be proved by induction)

- Calculate successive powers of $A$:

$$A^2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 2 & 1 & 1 & 0 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 2 & 1 \\
1 & 0 & 1 & 1 & 3 \\
\end{bmatrix}, \quad A^3 = \begin{bmatrix}
0 & 2 & 1 & 1 & 0 \\
2 & 0 & 1 & 1 & 4 \\
1 & 1 & 2 & 3 & 4 \\
1 & 1 & 3 & 2 & 4 \\
0 & 4 & 4 & 4 & 2 \\
\end{bmatrix}, \quad \text{etc.}$$

Ans1: $k = 3$, Ans2: entry $(3, 5)$ of $A^3$ is 4.

The problem here is quite simple, however adjacency matrices can be useful for more complex systems.

References: *Linear Algebra with Applications* (S.J. Leon), *Applications of Linear Algebra* (H. Anton)