2 VECTOR SPACES AND SUBSPACES

Many are familiar with the concept of a vector as:

- something which has magnitude and direction. eg. force, velocity.
- an ordered pair or triple. eg. $(1, 3)$, $(x_1, y_1, z_1)$.

Vectors belong to what is known as a vector space. We now turn to a vector space called $\mathbb{R}^n$.

2.1 The Vector Space $\mathbb{R}^n$

Definitions

- If $n$ is a positive integer, then an ordered $n$-tuple is a sequence of $n$ real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered $n$-tuples is called $n$-space and is denoted by $\mathbb{R}^n$.

Examples

1. When $n = 1$ each ordered $n$-tuple consists of one real number; i.e. $\mathbb{R}$ is the set of real numbers.

2. When $n = 2$ ($\mathbb{R}^2$) we get the set of all 2-tuples, aka ordered pairs. This set can be geometrically represented by the cartesian $x - y$ plane.

3. Similarly, the vector space $\mathbb{R}^3$ is the set of ordered triples, which describe all points and directed line segments in 3-D space.

Definitions

- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ are equal iff $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$. 

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The sum of 2 vectors \( \mathbf{u} \) and \( \mathbf{v} \) is defined by

\[ \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \]

Let \( k \) be any scalar, then the scalar multiple \( k\mathbf{u} \) is defined by

\[ k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n) \]

The two operations of addition and scalar multiplication are called the standard operations on \( \mathbb{R}^n \).

The zero vector in \( \mathbb{R}^n \) is denoted by \( \mathbf{0} \) and is defined to be the vector

\[ \mathbf{0} = (0, 0, \ldots, 0) \]

The negative (or additive inverse) of \( \mathbf{u} \) is denoted by \( -\mathbf{u} \) and is defined by

\[ -\mathbf{u} = (-u_1, -u_2, \ldots, -u_n) \]

The difference of vectors in \( \mathbb{R}^n \) is defined by

\[ \mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u}) \]

The most important arithmetic properties of addition and scalar multiplication of vectors in \( \mathbb{R}^n \) are listed in the following theorem.

**Theorem 2.1** If \( \mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n), \) and \( \mathbf{w} = (w_1, w_2, \ldots, w_n) \) are vectors in \( \mathbb{R}^n \) and \( k \) and \( l \) are scalars, then:

1. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
2. \( \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \)
3. \( \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \)
4. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0}; \) that is, \( \mathbf{u} - \mathbf{u} = \mathbf{0} \)
5. \( k(lu) = (kl)u \)

6. \( k(u + v) = ku + kv \)

7. \( (k + l)u = ku + lu \)

8. \( 1u = u \)

The above definitions and properties relating to the vector space \( \mathbb{R}^n \), can be compared to the axioms of a general vector space, as we shall see in the next section.

### 2.2 Generalized Vector Spaces

#### Definitions

Let \( V \) be an arbitrary nonempty set of objects on which two operations are defined, addition and multiplication by scalars.

Note: these operations are not necessarily defined in the same way as the standard operations of addition and scalar multiplication in the vector space \( \mathbb{R}^n \)! In fact, our only restriction on the way these operations are defined (in the general case) is as follows:

- If the axioms below are satisfied by all objects \( u, v, w \) in \( V \) and all scalars \( k \) and \( l \), then \( V \) is called a **vector space** and the objects in \( V \) are called **vectors**.

#### Vector Space Axioms

1. \( u + v \in V, \quad \forall u, v \in V \)

2. \( u + v = v + u \)

3. \( u + (v + w) = (u + v) + w \)

4. There exists a **zero vector** \( 0 \) in \( V \), defined such that

\[ u + 0 = 0 + u = u, \quad \forall u \in V \]

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5. There exists an object $-u \in V$, called the **negative** of $u$, such that

$$u + (-u) = -u + u = 0, \quad \forall u \in V$$

6. If $k$ is any scalar and $u \in V$, then $k u \in V$

7. $k(lu) = (kl)u$

8. $k(u + v) = ku + kv$

9. $(k + l)u = ku + lu$

10. $1u = u$

**Remark**

Scalars may be either real numbers or complex numbers. Vector spaces in which the scalars are real are referred to as Real Vector Spaces. Discussion in this subject will be limited to Real Vector Spaces.

**Examples of Vector Spaces**

A wide variety of vector spaces are possible under the above definition as illustrated by the following examples.

1. The set $V$ for all $2 \times 2$ matrices with real entries is a vector space if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication.

2. Let $V$ be the set of real-valued functions defined on the entire real line $(-\infty, \infty)$. If $f$ and $g$ are two such functions and $k$ is any real number, define the sum $f + g$ and the scalar multiple $kf$, respectively, by

$$ (f + g)(x) = f(x) + g(x) $$

$$ (kf)(x) = kf(x) $$

Show that $V$ is a vector space.
Proof:
Axiom 1: If \( f, g \in V \) show that \( f + g \in V \) also, i.e. show that the sum is a real-valued function defined on the entire real line. By definition
\[
\begin{align*}
f(x) &\in \mathbb{R} \quad \forall \quad x \in (-\infty, \infty) \\
g(x) &\in \mathbb{R} \quad \forall \quad x \in (-\infty, \infty),
\end{align*}
\]
therefore
\[
(f + g)(x) = f(x) + g(x) \in \mathbb{R} \quad \forall \quad x \in (-\infty, \infty).
\]
Axiom 6: Similarly, for any real number \( k \) we have
\[
(kf)(x) = kf(x) \in \mathbb{R} \quad \forall \quad x \in (-\infty, \infty).
\]
Axiom 2:
\[
(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)
\]
\[
\forall \quad x \in (-\infty, \infty).
\]
Axiom 3:
\[
(f + (g + h))(x) = f(x) + (g(x) + h(x)) = f(x) + g(x) + h(x)
\]
\[
= (f(x) + g(x)) + h(x) = ((f + g) + h)(x)
\]
\[
\forall \quad x \in (-\infty, \infty).
\]
Axiom 4:
Let \( 0 \in V \) be the constant function that is identically zero for all values of \( x \).
Then
\[
(f + 0)(x) = f(x) + 0 = f(x) = 0 + f(x) = (0 + f)(x)
\]
\[
\forall \quad x \in (-\infty, \infty).
\]
Axiom 5:
Let the negative of \( f \) be \(-f\). Then
\[
(f + (-f))(x) = f(x) + (-f(x)) = 0(x) = -f(x) + f(x) = (-f + f)(x)
\]
\[\forall \ x \in (-\infty, \infty).\]

Axiom 7:
\[
(k(lf))(x) = klf(x) = (kl)f(x) = ((kl)f)(x)
\]
\[\forall \ x \in (-\infty, \infty)\]

Axiom 8:
\[
(k(f + g))(x) = kf(x) + kg(x) = (kf)(x) + (kg)(x)
\]
\[\forall \ x \in (-\infty, \infty)\]

Axiom 9:
\[
((k + l)f)(x) = kf(x) + lf(x) = (kf)(x) + (kl)(x) = (k + l)f)(x)
\]
\[\forall \ x \in (-\infty, \infty).\]

Axiom 10:
\[
(1f)(x) = 1f(x) = f(x)
\]
\[\forall \ x \in (-\infty, \infty)\]

3. Let \( V = \mathbb{R}^2 \), with two arbitrary vectors \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \), and let \( k \) be any real number. The addition and scalar multiplication operations are defined (respectively):
\[
\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)
\]
\[
k \mathbf{u} = (ku_1, 0)
\]
Note that the scalar multiplication operation is not the standard one! In this example, $V$ is not a vector space under these operations.

### 2.2.1 Some Properties of Vectors

All vector spaces obey the following theorem.

**Theorem 2.2** Let $V$ be a vector space, $u$ a vector in $V$, and $k$ a scalar; then:

(a) $0u = 0$

(b) $k0 = 0$

(c) $(-1)u = -u$

(d) If $ku = 0$, then $k = 0$ or $u = 0$.

The proof follows from the Vector Space Axioms.

### 2.3 Subspaces

**Definitions**

- A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and scalar multiplication defined on $V$.

In general, all ten vector space axioms must be verified to show that a set $W$, with defined addition and scalar multiplication, forms a vector space. However, if $W$ is part of a larger set $V$ that is already known to be a vector space, then certain axioms need not be verified for $W$ because they are “inherited” from $V$. For example, there is no need to check that $u + v = v + u$ (Axiom 2) holds for $W$ because this holds for all vectors in $V$ and consequently holds for all vectors in $W$. For a subspace, the vector space axioms reduce to the following theorem:
Theorem 2.3  If \( W \) is a set of one or more vectors from a vector space \( V \), then \( W \) is a subspace of \( V \) if and only if the following conditions hold.

(a) If \( u \) and \( v \) are vectors in \( W \), then \( u + v \) is in \( W \).

(b) If \( k \) is any scalar and \( u \) is any vector in \( W \), then \( ku \) is in \( W \).

Proof:  If \( W \) is a subspace of \( V \), then all the vector space axioms are satisfied; in particular, Axioms 1 and 6 hold. But these are precisely conditions (a) and (b).

Conversely, assume conditions (a) and (b) hold. Since these conditions are vector space Axioms 1 and 6, it only remains to be shown that \( W \) satisfies the remaining eight axioms. Axioms 2, 3, 7, 8, 9 and 10 are automatically satisfied by the vectors in \( W \) since they are satisfied by all vectors in \( V \). Therefore, to complete the proof, we need only verify that Axioms 4 and 5 are satisfied by vectors in \( W \).

Let \( u \) be any vector in \( W \). By condition (b), \( ku \) is in \( W \) for every scalar \( k \). Setting \( k = 0 \), it follows from Theorem 2.2 that \( 0u = 0 \) is in \( W \), and setting \( k = -1 \), it follows that \((-1)u = -u \) is in \( W \).

Remarks

- Note that a consequence of (b) is that \( 0 \) is an element of \( W \).

- A set \( W \) of one or more vectors from a vector space \( V \) is said to be closed under addition if condition (a) in Theorem 2.3 holds and closed under scalar multiplication if condition (b) holds. Thus, Theorem 2.3 states that \( W \) is a subspace of \( V \) if and only if \( W \) is closed under addition and closed under scalar multiplication.

Examples of Subspaces

1. A plane through the origin of \( \mathbb{R}^3 \) forms a subspace of \( \mathbb{R}^3 \). This is evident geometrically as follows: Let \( W \) be any plane through the origin and let \( u \) and \( v \) be any vectors in \( W \) other than the zero vector. Then \( u + v \) must lie in \( W \)
because it is the diagonal of the parallelogram determined by \( u \) and \( v \), and \( ku \) must lie in \( W \) for any scalar \( k \) because \( ku \) lies on a line through \( u \). Thus, \( W \) is closed under addition and scalar multiplication, so it is a subspace of \( \mathbb{R}^3 \).

2. A line through the origin of \( \mathbb{R}^3 \) is also a subspace of \( \mathbb{R}^3 \). It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well. Thus, \( W \) is closed under addition and scalar multiplication, so it is a subspace of \( \mathbb{R}^3 \).

3. Let \( n \) be a positive integer, and let \( W \) consist of all functions expressible in the form

\[
p(x) = a_0 + a_1 x + \cdots + a_n x^n
\]

where \( a_0, \ldots, a_n \) are real numbers. Thus, \( W \) consists of the zero function together with all real polynomials of degree \( n \) or less. The set \( W \) is a subspace of the vector space of all real-valued functions discussed in Example 2 of section 2.2.

### 2.4 Linear Combination of Vectors

**Definition**

- A vector \( w \) is called a **linear combination** of the vectors \( v_1, v_2, \ldots, v_r \) if it can be expressed in the form

\[
w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r
\]

where \( k_1, k_2, \ldots, k_r \) are scalars.

**Example**

Consider the vectors \( u = (1, 2, -1) \) and \( v = (6, 4, 2) \) in \( \mathbb{R}^3 \). Show that \( w = (9, 2, 7) \) is a linear combination of \( u \) and \( v \) and that \( w' = (4, -1, 8) \) is not a linear combination of \( u \) and \( v \).
2.4.1 Spanning

**Theorem 2.4** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in a vector space $V$, then:

(a) The set $W$ of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is a subspace of $V$.

(b) $W$ is the smallest subspace of $V$ that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$; every other subspace of $V$ that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ must contain $W$.

**Proof:**

(a) To show that $W$ is a subspace of $V$, it must be shown that it is closed under addition and scalar multiplication. There is at least one vector in $W$, namely, $\mathbf{0}$, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r$. Now let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors in $W$:

\[
\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r
\]

\[
\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r
\]

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore

\[
\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \cdots + (c_r + k_r)\mathbf{v}_r
\]

and, for any scalar $k$,

\[
k\mathbf{u} = (kc_1)\mathbf{v}_1 + (kc_2)\mathbf{v}_2 + \cdots + (kc_r)\mathbf{v}_r
\]

Thus, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ and consequently lie in $W$. Therefore, $W$ is closed under addition and scalar multiplication.

(b) Each vector $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_1 \ldots, \mathbf{v}_r$ since we can write

\[
\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_r
\]

Therefore, the subspace of $W$ contains each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let $W'$ be any other subspace that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Since $W'$ is closed under addition and scalar multiplication, it must contain all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Thus $W'$ contains each vector of $W$. \qed

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Definitions

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in a vector space $V$, then the subspace $W$ of $V$ consisting of all linear combinations of the vectors in $S$ is called the **space spanned** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, and it is said that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ **span** $W$. The following notation is used:

$$W = \text{span}(S) \quad \text{or} \quad W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

Example

1. The polynomials $1, x, x^2, \ldots, x^n$ span the vector space $P_n$ (defined previously) since each polynomial $p$ in $P_n$ can be written as a linear combination of $1, x, x^2, \ldots, x^n$:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

This can be denoted by writing

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$

Spanning sets are not unique. For example, any two noncollinear vectors that lie in the $x - y$ plane will span the $x - y$ plane. Also, any nonzero vector on a line will span the same line.

**Theorem 2.5** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are two sets of vectors in a vector space $V$, then

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in $S$ is a linear combination of those in $S'$, and each vector in $S'$ is a linear combination of those in $S$.

**Proof:** Assume each vector in $S$ is a linear combination of those in $S'$. Then

$$\text{span}(S) \subseteq \text{span}(S')$$

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Also, if each vector in $S'$ is a linear combination of those in $S$, then

$$\text{span}(S') \subseteq \text{span}(S)$$

and therefore

$$\text{span}(S) = \text{span}(S').$$

Conversely, assume there is a vector $v_1 \in S$ that is not a linear combination of the vectors in $S'$:

$$v_1 \neq a_1w_1 + a_2w_2 + \cdots + a_nw_n$$

for all possible $a_1, a_2, \ldots, a_n$. Then

$$v_1 \in \text{span}(S) \text{ but } v_1 \notin \text{span}(S').$$

Therefore

$$\text{span}(S) \neq \text{span}(S')$$

and similarly for a vector $w_j \in S'$ that is not a linear combination of the vectors in $S$.

\[\blacksquare\]

### 2.5 Linear Independence

Recall that a set of vectors $S$ spans a vector space $V$ if every vector in $V$ is expressible as a linear combination of the vectors in $S$. In general, it is possible that there may be more than one way to express a vector in $V$ as a linear combination of vectors in a spanning set. This section will focus on the conditions under which each vector in $V$ is expressible as a unique linear combination of the spanning vectors. Spanning sets with this property play a fundamental role in the study of vector spaces.

**Definitions**

- If $S = \{v_1, v_2, \ldots, v_r\}$ is a nonempty set of vectors, then the vector equation

  $$k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0$$

  is called the **linear dependence condition**. If this equation has only the trivial solution $k_1 = k_2 = \cdots = k_r = 0$, then we say that the vectors $v_1, v_2, \ldots, v_r$ are **linearly independent**. If there exists a nontrivial solution, then the vectors are **linearly dependent**.
has at least one solution, namely

\[ k_1 = 0, k_2 = 0, \ldots, k_r = 0 \]

If this is the only solution, then \( S \) is called a **linearly independent** set. If there are other solutions, then \( S \) is called a **linearly dependent** set.

**Examples**

1. If \( \mathbf{v}_1 = (2, -1, 0, 3), \mathbf{v}_2 = (1, 2, 5, -1) \) and \( \mathbf{v}_3 = (7, -1, 5, 8) \), then the set of vectors \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is linearly dependent, since \( 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0 \).

2. The polynomials

\[
p_1(x) = 1 - x, \quad p_2(x) = 5 + 3x - 2x^2 \quad \text{and} \quad p_3(x) = 1 + 3x - x^2
\]

form a linearly dependent set in \( P_2 \) since \( 3p_1(x) - p_2(x) + 2p_3(x) = 0 \).

3. Consider the vectors \( \mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1) \) in \( \mathbb{R}^3 \). In terms of components the vector equation \( k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0} \) becomes

\[
k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)
\]

\[
\implies (k_1, k_2, k_3) = (0, 0, 0).
\]

Thus the set \( S = \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \} \) is linearly independent.

The following two theorems follow quite simply from the definition of linear independence and linear dependence.

**Theorem 2.6** A set \( S \) with two or more vectors is:

(a) **Linearly dependent if and only if** at least one of the vectors in \( S \) is expressible as a linear combination of the other vectors in \( S \);

(b) **Linearly independent if and only if** no vector in \( S \) is expressible as a linear combination of the other vectors in \( S \).
Example

1. Recall that the vectors
\[ \mathbf{v}_1 = (2, -1, 0, 3), \mathbf{v}_2 = (1, 2, 5, -1) \text{ and } \mathbf{v}_3 = (7, -1, 5, 8) \]

were linearly dependent because
\[ 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0} \]

It is obvious from the equation that
\[ \mathbf{v}_1 = \frac{-1}{3} \mathbf{v}_2 + \frac{1}{3} \mathbf{v}_3, \quad \mathbf{v}_2 = -3\mathbf{v}_1 + 1\mathbf{v}_3, \quad \mathbf{v}_3 = 3\mathbf{v}_1 + 1\mathbf{v}_2 \]

**Theorem 2.7** (a) A finite set of vectors that contains the zero vector is linearly dependent.

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

2.6 Operations on Vector Spaces

Definitions

- The addition of two vector spaces is defined by: \( U + V = \{u + v | u \in U, v \in V\} \)
- The intersection \( \cap \) of two vector spaces is defined by:
\[ U \cap V = \{w | w \in U \text{ and } w \in V\} \]