MATH2300 Assignment One

Due Tuesday August 10th at 5pm (Hand in to Room 67-448)

1. (a) Consider the system of equations where $a, b, c \in \Re$:

$$\begin{array}{rcrcrl} x & + & y & + & 2z = & a \\ x & & & + & z = & b \\ 2x & + & y & + & 3z = & c \end{array}$$

Show that in order for this system to be consistent, a, b and c must satisfy $c = a + b$. Now suppose that the system is homogeneous. Give a basis for the solution space and state its dimension.

(b) For which values of $a \in \Re$ will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{array}{rcrcrl} x & + & 2y & + & 3z = & 4 \\ 3x & + & 8y & - & z = & 2 \\ 4x & + & 10y & + & (a^2 - 23)z = & a + 1 \end{array}$$

2. (a) Let V be a real vector space with addition $+$ and scalar multiplication $\cdot$ and let $\mathbf{p}$ be a fixed vector in V. Define a new addition and scalar multiplication on V by the formulae

$$\begin{aligned} \mathbf{u} \oplus \mathbf{v} &= \mathbf{u} + \mathbf{v} + \mathbf{p} \\ t \otimes \mathbf{v} &= t \cdot \mathbf{v} + (t - 1) \cdot \mathbf{p} \end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in V$, $t \in \Re$. Show that V is also a vector space under the new operations. Explicitly state the zero vector and the inverse of an arbitrary vector $\mathbf{w} \in V$.

(b) Define a new vector addition on $\Re^2$ by

$$(x_1, y_1) \oplus (x_2, y_2) = (\sqrt{x_1{}^2 + x_2{}^2}, \sqrt{y_1{}^2 + y_2{}^2})$$

Alas! $\Re^2$ with this addition and the standard scalar multiplication is NOT a vector space. Which axiom fails? Provide a counterexample demonstrating this.

3. (a) If U and V are subspaces of a vector space W, prove that the subset of W defined by $U + V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of W.

(b) If U_1 and U_2 are subspaces of a vector space V and $U_1 \cup U_2$ is also a subspace of V, prove that $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

4. Let $A \in M_{n\times n}(\Re)$ and let U be the subset of $M_{n\times n}(\Re)$ defined by

$$U = \{X \in M_{n\times n}(\Re) \mid AX = XA\}$$

(a) Prove that U is a subspace of $M_{n\times n}(\Re)$.

(b) Find a basis for U when

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad \lambda \neq \mu$

Hint: You may assume the following facts:
(i) Matrix multiplication is associative.
(ii) Matrix multiplication is left and right distributive over matrix addition.
(iii) If α is a scalar and A and B are matrices of the appropriate sizes, then $(\alpha A)B = A(\alpha B) = \alpha(AB)$.

5. (a) Determine the dimensions of the following subspaces of $\Re^4$.
(i) All vectors of the form $(a, b, c, 0)$
(ii) All vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$
(iii) All vectors of the form (a, b, c, d) where $a = b = c = d$

(b) Show that the vector space of all real-valued functions defined on the entire real line is infinite-dimensional. (Hint: Assume it is finite-dimensional with dimension n, and obtain a contradiction by producing $n+1$ linearly independent vectors.)

6. (a) Let V be the vector space P_2. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1+x, 1+x^2, x+x^2\}$ be two bases for V. Calculate the change of basis matrix from β to γ. Use this to calculate $[3 + x + -2x^2]_\gamma$.

(b) Let β be a basis for an n-dimensional real vector space V. Show that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ form a linearly independent set of vectors in V, then the coordinate vectors $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_r}]_\beta$ form a linearly independent set in $\Re^n$. Similarly, show that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ span V, then the coordinate vectors $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_r}]_\beta$ span $\Re^n$. Conclude that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ form a basis for V if and only if $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_n}]_\beta$ form a basis for $\Re^n$.