MATH2300 Assignment Two

Due Friday August 27th at 5pm (Hand in to Room 67-448)

1. (a) Find an orthonormal basis for the subspace of $\Re^4$ spanned by

$$\mathbf{u_1} = (1,0,1,0)^T,\ \mathbf{u_2} = (0,0,1,1)^T,\ \mathbf{u_3} = (1,0,1,1)^T$$

   (b) Extend this to an orthonormal basis for $\Re^4$.

2. (a) (i) Find bases for the null space, row space and column space of the matrix

$$\begin{bmatrix} -3 & 5 & 1 & 2 \\ 7 & 2 & 0 & -4 \\ -8 & 3 & 1 & 6 \end{bmatrix}$$

   State the rank and nullity of A.
   (ii) Let $S = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}\}$ where

$$w_1 = \begin{bmatrix} -3 \\ 7 \\ -8 \end{bmatrix} \quad w_2 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \quad w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w_4 = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}.$$

   Find a subset of S that forms a basis for the space spanned by S. Express the vector or vectors in S which are not in this basis as a linear combination of the basis vectors.

   (b) PART B HAS BEEN REMOVED FROM THE ASSIGNMENT :)

3. (a) Let $A \in M_{3\times3}(\Re)$ such that $A^2 = 0$ and $A \neq 0$. Prove that

   (i) $C(A) \subset N(A)$.

   (ii) $\text{rank}(A) = 1$.

   (b) Now let $A \in M_{n\times n}(\Re)$ such that $A^2 = A$. Prove that

   (i) If $Y \in C(A)$, then $AY = Y$.

   (ii) $\Re^n = N(A) + C(A)$.

4. Let $T : U \to V$ be a linear transformation.

   (a) If $\text{Ker}(T) = \{\mathbf{0}\}$ and $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ are linearly independent in U, prove that $T(\mathbf{u_1}), \ldots, T(\mathbf{u_n})$ are linearly independent in V.

   (b) Prove that T is one-to one (ie. $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in U) if and only if $\text{Ker}(T) = \{\mathbf{0}\}$.

(c) Suppose that $\dim(U) = \dim(V)$. If $\mathrm{Ker}(T) = \{\mathbf{0}\}$, show that $\mathrm{Im}(T) = V$. Let A and B be non-singular $n \times n$ matrices over $\Re$ and let $V = M_{n\times n}(\Re)$. Let $X \in V$. Show that the mapping $T : V \to V$ defined by $T(X) = AXB$ has the property that $\mathrm{Ker}(T) = \{\mathbf{0}\}$ and $\mathrm{Im}(T) = V$.

5. A mapping $T : P_2(\Re) \to \Re^3$ is defined by

$$T(f(x)) = \begin{bmatrix} f(1) \\ f(0) \\ f(-1) \end{bmatrix}$$

(a) Prove that T is a linear transformation.

(b) If $S : \Re^3 \to P_2(\Re)$ is the linear transformation defined by

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = b + \frac{a-c}{2}x + \frac{a-2b+c}{2}x^2$$

verify that $S \circ T = I_{P_2(\Re)}$ and $T \circ S = I_{\Re^3}$.

6. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and let $T : M_{2\times 2}(\Re) \to M_{2\times 2}(\Re)$ be the linear transformation defined by $T(X) = AX - XA$. Find a basis for $\mathrm{Im}(T)$ and Ker (T). State $\mathrm{rank}(T)$ and $\mathrm{nullity}(T)$.