MATH2300 Assignment Three

Due Friday 10th September at 5pm (Hand in to Room 67-448)

1. Let U be a vector space with basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, and let V be a vector space with basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. Let $T : U \to V$ be the linear transformation defined by

$$\begin{aligned} T(\mathbf{u_1}) &= \mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3} - 2\mathbf{v_4} \\ T(\mathbf{u_2}) &= -2\mathbf{v_1} + 5\mathbf{v_2} - 13\mathbf{v_3} - 2\mathbf{v_4} \\ T(\mathbf{u_3}) &= 2\mathbf{v_1} + \mathbf{v_2} + 3\mathbf{v_3} - 2\mathbf{v_4} \end{aligned}$$

   (a) Find the matrix of transformation relative to the bases β and γ, $[T]_\beta^\gamma$.

   (b) Use this matrix to find bases for the kernel and image of T. State the rank and nullity of T.

2. Let $T : P_2[\Re] \to P_2[\Re]$ be the mapping defined by

$$T(f(x)) = f'(x)g(x) + 2f(x)$$

where $g(x) = 3 + x$ and $f'(x)$ is the formal derivative of $f(x)$. (That is, if $f = a_0 + a_1 x + a_2 x^2$, then $f'(x) = a_1 + 2a_2 x$, where $a_0, a_1, a_2 \in \Re$). Let $S : P_2[\Re] \to \Re^3$ be the linear transformation defined by

$$S(a + bx + cx^2) = \begin{bmatrix} a+b \\ c \\ a-b \end{bmatrix}$$

where $a, b, c \in \Re$. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard bases for $P_2[\Re]$ and $\Re^3$ respectively.

   (a) Prove that T is a linear transformation.

   (b) Find $[S]_\beta^\gamma$, $[T]_\beta^\beta$ and $[S \circ T]_\beta^\gamma$ and verify that $[S \circ T]_\beta^\gamma = [S]_\beta^\gamma [T]_\beta^\beta$.

3. (a) Prove that for any positive integer k, if A and B are similar matrices, then A^k and B^k are also similar.

   (b) Let A and B be $n \times n$ matrices with real entries. Prove that the relation "A is similar to B" is an equivalence relation on $M_{n\times n}(\Re)$. That is, the relation is reflexive, symmetric and transitive.

   (c) If λ is an eigenvalue of A, prove that λ^2 is an eigenvalue of A^2.

4. Let $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

(a) Find the eigenvalues of A and state the algebraic multiplicity of each eigenvalue.

(b) Find bases for the eigenspaces corresponding to each eigenvalue and state the geometric multiplicity of each eigenvalue. Explain why A is diagonalizable.

(c) Find a non-singular matrix $P \in M_{3\times3}(\Re)$ that diagonalizes A and determine $P^{-1}AP$.

5. Let

$$A = \begin{bmatrix} 1 & -4 & 0 \\ -4 & 3 & -4 \\ 0 & -4 & 5 \end{bmatrix}$$

Find an orthogonal matrix P that diagonalizes A. Determine $P^T AP$.

6. (a) QUESTION 6(a) HAS BEEN REMOVED FROM THE ASSIGNMENT

(b) Evaluate A^m where

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$