1   BASICS OF LINEAR ALGEBRA

Quantum physics, mechanics, Wall Street and environmental management - what do all these things have in common? The answer, of course, is **systems of linear equations**. That's right folks, whether you're looking at the Heisenberg Uncertainty Principle or controlling the amount of waste that pollutes our oceans, you're going to be interested in the study of systems of linear equations. In this section some basic terminology will be introduced, along with a method for solving such systems.

1.1   Basic Definitions

Definitions

- A **linear equation in n variables** $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form
$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$
where $a_1, a_2, \ldots, a_n$, and b are real constants. The variables in a linear equation are sometimes called the **unknowns**.

- An arbitrary **system of m linear equations in n unknowns** is one that can be written as
$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$
where once again $x_1, x_2, \cdots, x_n$ are the unknowns and the subscripted a's and b's denote constants.

- A system of linear equations is said to be **homogenous** if the constant terms are all zero; that is, the system has the form:

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0$$

- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a **solution** of the system if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution of every equation in the system.

Example

The following equations constitute a system of 2 linear equations in 3 unknowns.

$$4x_1 - x_2 + 3x_3 = -1$$

$$3x_1 + x_2 + 9x_3 = -4$$

The system has the solution $x_1 = 1, x_2 = 2, x_3 = -1$ since these values satisfy both equations. The set of values $x_1 = 1, x_2 = 8, x_3 = 1$ is not a solution since these values satisfy only the first of the two equations.

1.2 Solving a System of Linear Equations

1.2.1 The Augmented Matrix

Remembering that an arbitrary system of m linear equations in n unknowns can be written as:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

If one mentally keeps track of the +'s, the x's, and the ='s, then a system of m linear equations in n unknowns can be completely described by writing only the rectangular array of constants:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Such a matrix is referred to as the **augmented matrix** of the system.

Remark

When constructing an augmented matrix, the unknowns must appear in the same order in each equation and the constants must be on the right.

1.2.2 Elementary Row Operations

As you perhaps have discovered in (numerous!) previous subjects, the basic method for solving a system of linear equations is to replace the given system with a set that has the same solution set but is easier to solve. By applying the below three types of operation, unknowns are eliminated systematically without altering the solution set.

1. Multiply an equation through by a nonzero constant.

2. Interchange two equations.

3. Add a multiple of one equation to another.

Since the rows of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix. These operations are called **elementary row operations**.

Definitions

- The three **elementary row operations** are:

  1. Multiplying row i by a non-zero constant t:

  $$R_i \to tR_i$$

  2. Interchange rows i and j:
  $$R_i \leftrightarrow R_j$$

  3. Adding t times row i to row j:

  $$R_j \to R_j + tR_i$$

- Matrix A is **row-equivalent** to matrix B if B is obtained from A by a sequence of elementary row operations. Clearly if B is row-equivalent to A then A is row-equivalent to B.

Remark
Elementary row operations can be used to convert an augmented matrix into a special form, called the reduced row-echelon form, without changing the solution set. Once the conversion is complete the solution set is easily obtained.

1.2.3   Reduced Row-Echelon Form

Definitions

- A matrix is in **reduced row-echelon** form or **rref** for short when the following four conditions are satisfied:

  1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. (This is called a **leading 1.**)

  2. If there are any rows that consists entirely of zeros, then they are grouped together at the bottom of the matrix.

  3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

  4. Each column that contains a leading 1 has zeros everywhere else.

- A matrix having properties **1, 2** and **3** (but not necessarily 4) is said to be in **row echelon form** or **ref** for short.

Examples

- The following matrices are in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following matrices are in row-echelon form:

$$\begin{bmatrix} 1 & 4 & 3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark It can be shown that every matrix has a unique reduced row-echelon form.

1.2.4 Gauss Jordan Algorithm

Let A be the augmented matrix that completely describes a given system of m linear equations in n unknowns. It has already been stated that the solution set of this system can be easily obtained from the rref of A. The Gauss Jordan Algorithm converts A to its (unique) reduced row-echelon form using elementary row operations. The procedure is outlined below. As each step is stated, the idea will be illustrated by reducing the following matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

1. Locate the left most column that does not consist entirely of zeros. In the case above this is column one.

2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $\frac{1}{a}$ in order to introduce a leading 1.

$$R_1 \to \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$R_3 \to R_3 + -2R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form. In this case the next leftmost nonzero column is column three.

$$R_2 \to \frac{-1}{2}R_2 \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$R_3 \to R_3 + -5R_2 \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$R_3 \to 2R_3 \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The entire matrix is now in row-echelon form. To find the reduced row-echelon form the following step is needed.

6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_2 \to R_2 + \frac{7}{2}R_3 \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 \to R_1 + -6R_3 \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 \to R_1 + 5R_2 \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

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1.3 Consistent and Inconsistent Systems

Definitions

- A system of linear equations that has no solution is said to be **inconsistent**.

- A system of linear equations with at least one solution is said to be **consistent**.

1.3.1 Demonstration of Inconsistent and Consistent Systems in the $x - y$ Plane

To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns x and y:

$$a_1 x + b_1 y = c_1 \rightarrow l_1$$

$$a_2 x + b_2 y = c_2 \rightarrow l_2$$

(NB a_1 and b_1 not both zero and a_2 and b_2 not both zero). The graphs of these equations are lines: call them l_1 and l_2. Since a point (x, y) lies on a line if and only if the numbers x and y satisfy the equation of the line, the solutions of the system of equations correspond to points of intersection of l_1 and l_2. There are three possibilities:

- The lines l_1 and l_2 may be parallel, in which case there is no intersection and consequently no solution to the system. Thus the system of equations is said to be **inconsistent**.

- The lines l_1 and l_2 may intersect at only one point, in which case the system has exactly one solution. Thus the system of equations is said to be **consistent**.

- The lines l_1 and l_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system. Thus this system is also said to be **consistent**.

Although we have only demonstrated this for a special case, it can be shown in general that the following theorem holds.

Theorem 1.1. *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.*

1.4   Solving Homogenous Systems of Linear Equations

We solve the homogenous system of linear equations below:

$$
\begin{array}{rrrrrr}
2x_1+ & 2x_2- & x_3 & & +x_5 = & 0 \\
-x_1- & x_2+ & 2x_3- & 3x_4+ & x_5 = & 0 \\
x_1+ & x_2- & 2x_3 & & -x_5 = & 0 \\
& & x_3+ & x_4+ & x_5 = & 0
\end{array}
$$

The augmented matrix for the system is:

$$
\left[\begin{array}{rrrrrr}
2 & 2 & -1 & 0 & 1 & 0 \\
-1 & -1 & 2 & -3 & 1 & 0 \\
1 & 1 & -2 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0
\end{array}\right]
$$

Reducing the matrix to rref, we obtain:

$$
\left[\begin{array}{rrrrrr}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

The corresponding system of equations is:

$$
\begin{array}{rrrr}
x_1+ & x_2 & +x_5 = & 0 \\
& x_3 & +x_5 = & 0 \\
& x_4 & = & 0
\end{array}
$$

Solve each equation for the leading variables.

$x_1 = -x_2 - x_5$

$x_3 = -x_5$

$x_4 = 0$

Then the general solution is

$$
x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t.
$$

Note that the trivial solution is obtained when $s = t = 0$.

   This example illustrates two important points about solving a homogenous system of linear equations:

- None of the three elementary row operations alter the final column of zero's. Therefore, the corresponding system of equations to the rref is also a homogenous system of linear equations.

- Depending on the number of zero rows in the rref the number of equations in the reduced system is less than or equal to the number of equations in the original system.

These two observations are instrumental in proving the following theorem.

Theorem 1.2. *A homogenous system of linear equations with more unknowns than equations has infinitely many solutions*

Proof. If a given homogenous system has m equations in n unknowns such that $m < n$, and if there are r non-zero rows in the rref of the augmented matrix, it follows that $r < n$. The system of equations corresponding to the rref of the augmented matrix will have the following form:

$$
\begin{array}{rl}
\cdots x_{k_1} \qquad\qquad\qquad +\sum(\ ) & = 0 \\
\cdots x_{k_2} \qquad\qquad +\sum(\ ) & = 0 \\
\cdots \ddots \qquad\qquad\quad \vdots & \\
x_{k_r} \ +\sum(\ ) & = 0
\end{array}
$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the leading variables and $\sum(\ )$ denotes sums (possibly all different) that involve the $n - r$ free variables. Solving for the leading variables gives

$$
x_{k_1} = -\sum(\ )
$$

$$
x_{k_2} = -\sum(\ )
$$

$$
\vdots
$$

$$
x_{k_r} = -\sum(\ )
$$

As in the above example, arbitrary values can be assigned to the free variables on the right hand side and thus infinitely many solutions are obtained for the system. $\square$

1.5   Important Matrices

We define certain matrices which are of special significance in linear algebra:

Definitions

- Square matrices taking the form of 1's on the main diagonal and 0's off the main diagonal, are called **identity matrices**. Such matrices are denoted by I. If it is important to emphasize the size, I_n shall be written to denote the $n \times n$ identity matrix. For instance I_3 is the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  If A is an $m \times n$ matrix, then

$$AI_n = A \;\; and \;\; I_m A = A$$

  Thus an identity matrix plays much the same role in matrix arithmetic as the number 1 plays in the numerical relationship $a \cdot 1 = 1 \cdot a = a$.

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be **invertible** or **non-singular** and B is called the **inverse** of A. If A is not invertible it is said to be **singular**.

- A square matrix in which all of the entries off the main diagonal are zero is called a **diagonal matrix**. Some examples are:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

  A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

1.5.1 Properties of Diagonal Matrices

Diagonal matrices enjoy the following properties.

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of the general form is

$$
D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}
$$

  Verify that $DD^{-1} = D^{-1}D = I$.

- Powers of diagonal matrices are easy to compute; once again it is left to the reader to verify that if D is a the general diagonal matrix and k is any positive integer, then

$$
D^k = \begin{bmatrix} d_1{}^k & 0 & \cdots & 0 \\ 0 & d_2{}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n{}^k \end{bmatrix}
$$

- A diagonal matrix is its own transpose. That is, $D = D^T$.

- To multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D. For example:

$$
\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}
$$

- To multiply a matrix A on the right by a diagonal matrix D, one can multiply successive columns of A by the successive diagonal entries of D. For example:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}
$$

2 VECTOR SPACES AND SUBSPACES

What is a vector? Many are familiar with the concept of a vector as:

- something which has magnitude and direction.

- an ordered pair or triple.

- a description for such quantities as force, velocity and acceleration.

Such vectors belong to the foundation vector space - $\Re^n$ - of all vector spaces. The properties of general vector spaces are based on the properties of $\Re^n$. It is therefore helpful to consider briefly the nature of $\Re^n$.

2.1 The Vector Space $\Re^n$

Definitions

- If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called **n-space** and is denoted by $\Re^n$.

When $n = 1$ each ordered n-tuple consists of one real number, and so $\Re$ may be viewed as the set of real numbers. Take $n = 2$ and one has the set of all 2-tuples which are more commonly known as **ordered pairs**. This set has the geometrical interpretation of describing all points and directed line segments in the Cartesian $x - y$ plane. The vector space $\Re^3$, likewise is the set of **ordered triples**, which describe all points and directed line segments in 3-D space.

In the study of 3-space, the symbol (a_1, a_2, a_3) has two different geometric interpretations: it can be interpreted as a point, in which case a_1, a_2 and a_3 are the coordinates, or it can be interpreted as a vector, in which case a_1, a_2 and a_3 are the components. It follows, therefore, that an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be viewed as a "generalized point" or a "generalized vector" - the distinction is mathematically unimportant. Thus, we can describe the 5-tuple $(1, 2, 3, 4, 5)$ either as a point or a vector in $\Re^5$.

Definitions

- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\Re^n$ are said to be **equal** if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$$

- The **sum u+v** is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

- Let k be any scalar, then the **scalar multiple $k\mathbf{u}$** is defined by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

- These two operations of addition and scalar multiplication are called the **standard operations** on $\Re^n$.

- The **zero vector** in $\Re^n$ is denoted by **0** and is defined to be the vector

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- The **negative** (or **additive inverse**) of u is denoted by $-u$ and is defined by

$$-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$$

- The **difference** of vectors in $\Re^n$ is defined by

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$$

The most important arithmetic properties of addition and scalar multiplication of vectors in $\Re^n$ are listed in the following theorem. This theorem enables us to manipulate vectors in $\Re^n$ without expressing the vectors in terms of components.

Theorem 2.1. *If* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, *and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *are vectors in* $\Re^n$ *and* k *and* l *are scalars, then:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; *that is,* $\mathbf{u} - \mathbf{u} = \mathbf{0}$

5. $k(l\mathbf{u}) = (kl)\mathbf{u}$

6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

7. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

8. $1\mathbf{u} = \mathbf{u}$.

2.2 Generalized Vector Spaces

The time has now come to generalize the concept of a vector. In this section a set of axioms are stated, which if satisfied by a class of objects, entitles those objects to be called "vectors". The axioms were chosen by abstracting the most important properties (Theorem 2.1) of vectors in $\Re^n$; as a consequence, vectors in $\Re^n$ automatically satisfy these axioms. Thus, the new concept of a vector, includes many new kinds of vector without excluding the 'common vector'. The new types of vectors include, among other things, various kinds of matrices and functions. The work covered in this section provides a powerful tool for the extension of geometrical visualisation to a wide variety of important mathematical problems where geometric intuition is otherwise unavailable. Briefly stated, the concept is this: It is possible to visualize vectors in $\Re^2$ and $\Re^3$ geometrically as arrows, which enables the drawing of physical and mental pictures which aid the solution of the problem. Now the axioms used to create the new kinds of vectors are based on the properties of vectors in $\Re^2$ and $\Re^3$, and therefore have many of the familiar properties of vectors in $\Re^2$ and $\Re^3$. Consequently, the solving of problems involving the new kinds of vectors, say matrices or functions, may be aided by visualizing geometrically what the corresponding problem would be like in $\Re^2$ or $\Re^3$.

Definitions

- Let V be an arbitrary nonempty set of objects on which two operations are defined, addition and multiplication by scalars (numbers). **Addition** is a rule (not necessarily the standard rule) for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u} + \mathbf{v}$, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; **scalar multiplication** is a rule (not necessarily the standard rule) for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and l, then V is called a **vector space** and the objects in V are called **vectors**.

  1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.

  2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

  3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

  4. There is an object $\mathbf{0}$ in V, called a **zero vector** for V, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V.

5. For each **u** in V, there is an object -**u** in V, called the **negative** of **u**, such that $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$;

6. If k is any scalar and **u** is any object in V, then $k\mathbf{u}$ is in V.

7. $k(l\mathbf{u}) = (kl)\mathbf{u}$

8. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

9. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

10. $1\mathbf{u} = \mathbf{u}$

Remark Depending on the application, scalars may be either real numbers or complex numbers. Vector spaces in which the scalars are complex are referred to as complex vector spaces. Vector spaces in which the scalars must be real are referred to as real vector spaces.

It is important to remember that the definition of a vector space specifies neither the nature of the vectors nor the operations. It is possible for any kind of object to be a vector, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on $\Re^n$. The only requirement is that the ten vector space axioms be satisfied. The notations $\oplus$ and $\odot$ are used in the notes for vector addition and scalar multiplication to distinguish between these operations and the standard vector operations previously introduced.

Examples of Vector Spaces

A wide variety of mathematical objects and operations satisfy the definition of a vector space. In each of the following examples, a nonempty set V and two operations, addition and scalar multiplication, are specified. We then demonstrate that each of the ten axioms are satisfied.

1. Show that the set V of all 2×2 matrices with real entries is a vector space if vector addition is defined to be matrix addition and scalar multiplication is defined to be matrix scalar multiplication.

   In this example the axioms will be verified in the following order: 1, 6, 2, 3, 4, 5, 7, 8, 9 and 10. Let

   $$\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}.$$

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To prove axiom 1, it must be shown that $\mathbf{u} + \mathbf{v}$ is an object in V; that is, $\mathbf{u} + \mathbf{v}$ is a 2×2 matrix with real entries.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

Similarly, axiom 6 holds because for any real number k

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

so that $k\mathbf{u}$ is a 2×2 matrix with real entries and consequently is an object in V.

Proving axiom 2:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$= \begin{bmatrix} v_{11} + u_{11} & v_{12} + u_{12} \\ v_{21} + u_{21} & v_{22} + u_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

Proving axiom 3:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \left(\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \right)$$

$$= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} + w_{11} & v_{12} + w_{12} \\ v_{21} + w_{21} & v_{22} + w_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & u_{22} + v_{22} + w_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

$$= \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

To prove axiom 4, an object $\mathbf{0}$ must be found in V such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{0}$ for all $\mathbf{u}$ in V. Define $\mathbf{0}$ to be

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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With this definition

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and since axiom 2 holds, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

To prove axiom 5, it must be shown that for every object $\mathbf{u}$ in V there is a negative $-\mathbf{u}$ such that $\mathbf{u} + -\mathbf{u} = \mathbf{0} = -\mathbf{u} + \mathbf{u}$. Let the negative of $\mathbf{u}$ be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and from axiom 2 $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

Proving axiom 7:

$$k(l\mathbf{u}) = k\left(l \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = k \begin{bmatrix} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{bmatrix} = \begin{bmatrix} klu_{11} & klu_{12} \\ klu_{21} & klu_{22} \end{bmatrix}$$

$$= kl \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (kl)\mathbf{u}$$

Proving axiom 8:

$$k(\mathbf{u} + \mathbf{v}) = k\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) = k \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} + kv_{11} & ku_{12} + kv_{12} \\ ku_{21} + kv_{21} & ku_{22} + kv_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$

$$= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = k\mathbf{u} + k\mathbf{v}$$

Proving axiom 9:

$$(k + l)\mathbf{u} = (k + l) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} (k + l)u_{11} & (k + l)u_{12} \\ (k + l)u_{21} & (k + l)u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} + lu_{11} & ku_{12} + lu_{12} \\ ku_{21} + lu_{21} & ku_{22} + lu_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{bmatrix} = k\mathbf{u} + l\mathbf{u}$$

...and finally, axiom 10 is a simple computation

$$1\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Therefore the set of all real 2×2 matrices, with matrix addition and matrix scalar multiplication form a vector space.

2. Let V by the set of real-valued functions defined on the entire real line $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two such functions and k is any real number, define the sum function $\mathbf{f} + \mathbf{g}$ and the scalar multiple $k\mathbf{f}$ by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$

$$(k\mathbf{f})(x) = kf(x)$$

It is again convenient to verify the axioms in the following order: 1, 6, 2, 3, 4, 5, 7, 8, 9 and 10.

To prove axiom 1 it must be shown that $\mathbf{f} + \mathbf{g}$ is an object in V; that is it must be shown that $\mathbf{f} + \mathbf{g}$ is a real-valued function defined on the entire real line.

$$f(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

$$g(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

therefore
$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

In the same way axiom 6 holds because for any real number k we have

$$(k\mathbf{f})(x) = kf(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 2:

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) = g(x) + f(x) = (\mathbf{g} + \mathbf{f})(x) \quad \forall \quad x \in (-\infty, \infty)$$

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Proving axiom 3:

$$(\mathbf{f} + (\mathbf{g} + \mathbf{h}))(x) = f(x) + (g(x) + h(x)) = f(x) + g(x) + h(x)$$

$$= (f(x) + g(x)) + h(x) = ((\mathbf{f} + \mathbf{g}) + \mathbf{h})(x) \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 4:

Let the **0** vector be the constant function that is identically zero for all values of x.

$$(\mathbf{f} + \mathbf{0})(x) = f(x) + 0 = (\mathbf{0} + f)(x) = \mathbf{f} \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 5: Let the negative of **f** be -**f** = -f(x).

$$(\mathbf{f} + (-\mathbf{f}))(x) = f(x) + -f(x) = \mathbf{0} \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 7:

$$(k+l)\mathbf{f} = (k+l)f(x) = kf(x) + lf(x) = (kf)(x) + (kl)(x) = k\mathbf{f} + l\mathbf{f} \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 8:

$$k(l\mathbf{f}) = k(lf(x)) = klf(x) = (kl)f(x) = (kl)\mathbf{f} \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 9:

$$(k(\mathbf{f}+\mathbf{g}))(x) = k(f(x)+g(x)) = kf(x)+kg(x) = (k\mathbf{f})(x)+(k\mathbf{g})(x) \quad \forall \quad x \in (-\infty, \infty)$$

Proving axiom 10:

$$1\mathbf{f} = 1f(x) = f(x) = \mathbf{f} \quad \forall \quad x \in (-\infty, \infty)$$

3. Let $V = \Re^2$ and let addition be the standard addition on Re^2. If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, and if k is any real number, then define scalar multiplication to be:

$$k\mathbf{u} = (ku_1, 0)$$

It can be shown that the first nine vector axioms hold. However there are values of **u** for which axiom 10 fails to hold. For example, if $\mathbf{u} = (u_1, u_2)$ such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, V is not a vector space with the stated operations.

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2.3 Some Properties of Vectors

More examples of vector spaces will be added to the list as the course continues. The following theorem gives a useful set of vector properties, all of which may be easily deduced from the axioms.

Theorem 2.2. *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:*

  (a) $0\mathbf{u} = \mathbf{0}$

  (b) $k\mathbf{0} = \mathbf{0}$

  (c) $(-1)\mathbf{u} = -\mathbf{u}$

  (d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = 0$.

2.4 Subspaces

It is possible for one vector space to be contained within a larger vector space. This section will look closely at this important concept.
Definitions

- A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, all ten vector space axioms must be verified to show that a set W with addition and scalar multiplication forms a vector space. However, if W is part of a larger set V that is already known to be a vector space, then certain axioms need not be verified for W, because they are "inherited" from V. For example, there is no need to check that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (axiom 2) holds for W because this holds for all vectors in V and consequently holds for all vectors in W. Likewise, Axioms 3, 7, 8, 9 and 10 are inherited by W from V. Thus to show that W is a subspace of a vector space V (and hence that W is a vector space), only Axioms 1, 4, 5 and 6 need to be verified. The following theorem reduces this list even further by showing that even Axioms 4 and 5 can be dispensed with.

Theorem 2.3. *If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if W is closed under addition and scalar multiplication.*

Proof. If W is a subspace of V, then all the vector space axioms are satisfied. In particular, axioms 1 and 6 hold. (That is, W is closed under addition and scalar multiplication.)

Conversely, assume W is closed under addition and scalar multiplication. Since these conditions are vector space Axioms 1 and 6, it only remains to be shown that W satisfies the remaining eight axioms. Axioms 2, 3, 7, 8, 9 and 10 are automatically satisfied by the vectors in W since they are satisfied by all vectors in V. Therefore, to complete the proof, we need only verify that Axioms 4 and 5 are satisfied by vectors in W.

Let $\mathbf{u}$ be any vector in W. Since by assumption W is closed under scalar multiplication $k\mathbf{u}$ is in W for every scalar k. Setting $k = 0$, it follows from Theorem 2.2 that $0\mathbf{u} = \mathbf{0}$ is in W, and setting $k = -1$, it follows that $(-1)\mathbf{u} = -\mathbf{u}$ is in W. $\qquad\square$

Remark Note that $\mathbf{0}$ is an element of any subspace W.

Examples of Subspaces

1. A plane through the origin of $\Re^3$ forms a subspace of $\Re^3$. This is evident geometrically as follows: Let W be any plane through the origin and let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in W other than the zero vector. Then $\mathbf{u} + \mathbf{v}$ must lie in W because it is the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, and $k\mathbf{u}$ must lie in W for any scalar k because $k\mathbf{u}$ lies on a line through $\mathbf{u}$. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\Re^3$.

2. A line through the origin of $\Re^3$ is also a subspace of $\Re^3$. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\Re^3$.

3. Let n be a positive integer, and let W consist of all functions expressible in the form

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

where $a_0, \ldots, a_n$ are real numbers. Thus, W consists of the zero function together with all real polynomials of degree n or less. The set W is a subspace of the vector space of all real-valued functions discussed in Example 2 of Section 2.2. To see this, let $\mathbf{p}$ and $\mathbf{q}$ be the polynomials

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

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and
$$q(x) = b_0 + b_1 x + \ldots + b_n x^n$$

Then

$$(\mathbf{p} + \mathbf{q})(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$$

and

$$(k\mathbf{p})(x) = kp(x) = (ka_0) + (ka_1)x + \ldots + (ka_n)x^n$$

These functions have the form given above, so $\mathbf{p} + \mathbf{q}$ and $k\mathbf{p}$ lie in W. This vector space W shall be denoted by the symbol P_n.

2.5 Linear Combinations of Vectors

Definitions

- A vector $\mathbf{w}$ is a **linear combination** of the vectors $v_1, v_2, \ldots, v_r$ if it can be expressed in the form

$$\mathbf{w} = k_1 v_1 + k_2 v_2 + \ldots + k_r v_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Example

1. Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\Re^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

   In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $k_1 \mathbf{u} + k_2 \mathbf{v} = \mathbf{w}$; that is in matrix form,

$$\begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$$

   Forming the augmented matrix of the system gives

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$

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Finding the rref of the augmented matrix gives

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are $k_1 = -3$ and $k_2 = 1$ and so $\mathbf{w} = -3\mathbf{u} + 1\mathbf{v}$.

Similarly, for $\mathbf{w}'$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is in matrix form,

$$\begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$$

Forming the augmented matrix of the system gives

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix}$$

Finding the rref of the augmented matrix gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly from the last row of the rref the system is inconsistent and therefore, no such scalars k_1 or k_2 exist. Consequently $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

2.6   Spanning Sets

If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are vectors in a vector space V, then some vectors in V may be linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and others may not. The following theorem shows that if a set W is constructed consisting of all those vectors that are expressible as linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, then W forms a subspace of V.

Theorem 2.4. *If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are vectors in a vector space V, then:*

(a) *The set W of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ is a subspace of V.*

(b) W is the smallest subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Every other subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ must contain W.

Proof.   (a) To show that W is a subspace of V, we must show that it is closed under addition and scalar multiplication. There is at least one vector in W, namely $\mathbf{0}$, since $\mathbf{0} = 0\mathbf{v_1} + 0\mathbf{v_2} + \ldots + 0\mathbf{v_r}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_r\mathbf{v_r}$$

and

$$\mathbf{v} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_r\mathbf{v_r}$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v_1} + (c_2 + k_2)\mathbf{v_2} + \ldots + (c_r + k_r)\mathbf{v_r}$$

and, for any scalar k,

$$k\mathbf{u} = (kc_1)\mathbf{v_1} + (kc_2)\mathbf{v_2} + \ldots + (kc_r)\mathbf{v_r}$$

Thus, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and consequently lie in W.

(b) Each vector $\mathbf{v_i}$ for $i = 1, 2, \ldots, r$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ since we can write

$$\mathbf{v_i} = 0\mathbf{v_1} + 0\mathbf{v_2} + \ldots + 1\mathbf{v_i} + \ldots + 0\mathbf{v_r}$$

Therefore, the subspace W contains each of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Let W' be any other subspace that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Since W' is closed under addition and scalar multiplication, it must contain all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and thus W' contains all vectors in W.

$\square$

Definitions

- If S = $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the **space spanned** by $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, and the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are said to **span** W. To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ the following notation is used.

$$W = span(S) \quad or \quad W = span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$$

Example: The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n defined previously since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. Thus we write:

$$P_n = span\{1, x, x^2, \ldots, x^n\}$$

Spanning sets are not unique. For example, any two noncollinear vectors that lie in the $x - y$ plane will span the $x - y$ plane. Also, any nonzero vector on a line will span that line.

Theorem 2.5. *If* $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ *and* $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ *are two sets of vectors in a vector space* V, *then*

$$span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\} = span\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$$

if and only if each vector in S *is a linear combination of those in* S', *and each vector in* S' *is a linear combination of those in* S.

Proof. If each vector in S is a linear combination of those in S' then $span(S) \subseteq span(S')$. If it is also true that each vector in S' is a linear combination of those in S then $span(S') \subseteq span(S)$, and therefore

$$span(S) = span(S').$$

If there is some vector $\mathbf{v_i} \in S$ such that

$$\mathbf{v_i} \neq a_1 \mathbf{w_1} + a_2 \mathbf{w_2} + \cdots + a_n \mathbf{w_n}$$

for any scalars $a_1, a_2, \ldots, a_n$ then $\mathbf{v_i} \in span(S)$ but $\mathbf{v_i} \notin span(S')$ therefore $span(S) \neq span(S')$ and vice versa. $\square$

2.7   Linear Independence

In the previous section it was stated that a set of vectors S spans a given vector space V if every vector in V is expressible as a linear combination of the vectors in S. In general, it is possible that there may be more than one way to express a vector in V as a linear combination of vectors in a spanning set. This section will focus on the

conditions under which each vector in V is expressible as a unique linear combination of the spanning vectors. Spanning sets with this property play a fundamental role in the study of vector spaces.

Definitions

- If $S = \{v_1, v_2, \ldots, v_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r} = \mathbf{0}$$

  has at least one solution, namely

$$k_1 = 0, k_2 = 0, \ldots, k_r = 0.$$

  If this is the only solution, then S is called a **linearly independent** set. If there are other solutions, then S is called a **linearly dependent** set.

Examples

1. If $\mathbf{v_1} = (2, -1, 0, 3), v_2 = (1, 2, 5, -1)$ and $v_3 = (7, -1, 5, 8)$, then the set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent, since $3\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} = \mathbf{0}$.

2. The polynomials

$$\mathbf{p_1} = 1 - x, \mathbf{p_2} = 5 + 3x - 2x^2, \ and \ \mathbf{p_3} = 1 + 3x - x^2$$

   form a linearly dependent set in P_2 since $3\mathbf{p_1} - \mathbf{p_2} + 2\mathbf{p_3} = \mathbf{0}$

3. Consider the vectors $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$ in $\Re^3$. In terms of components the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

   becomes

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$$

   or equivalently,

$$(k_1, k_2, k_3) = (0, 0, 0)$$

   Thus the set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent. A similar argument can be used to extend S to a linear independent set in $\Re^n$.

The following two theorems follow easily from the definition of linear independence and linear dependence.

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Theorem 2.6. *A set S of two or more vectors is:*

(a) *linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

(b) *linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.*

Theorem 2.7. (a) *A finite set of vectors that contains the zero vector is linearly dependent.*

(b) *A set of exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

2.8   Operations on Vector Spaces

Definitions

- The addition of two vector spaces is defined by: $U + V = \{\mathbf{u} + \mathbf{v} \,|\, \mathbf{u} \in U, \mathbf{v} \in V\}$

- The intersection $\cap$ of two vector spaces is defined by:

$$U \cap V = \{\mathbf{w} \,|\, \mathbf{w} \in U \text{ and } \mathbf{w} \in V\}$$

3   BASIS AND DIMENSION

A line is thought of as 1-dimensional, a plane 2-dimensional, and surrounding space as 3-dimensional. This section will make this intuitive notion of dimension precise and extend it to general vector spaces.

3.1   Coordinate Systems of General Vector Spaces

A line is thought of as 1-dimensional because every point on that line can be specified by one coordinate. In the same way, a plane is thought of as 2-dimensional because every point on that plane can be specified by two coordinates, and so on. What defines this coordinate system? Coordinate systems are most commonly defined by coordinate axes. In the case of the plane the x and y axes are used most frequently, but there is also a way of specifying the coordinate system with vectors. This can be done by replacing each axis with a vector of length one that points in the positive direction of the axis. In the case of the $x - y$ plane, the x and y-axes are replaced by the well known unit vectors $\mathbf{i}$ and $\mathbf{j}$ respectively. Let O be the origin of the system and P be any point in the plane. The point P can be specified by the vector $\overline{OP}$. Every vector, $\overline{OP}$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$:

$$\overline{OP} = a\mathbf{i} + b\mathbf{j}$$

The coordinates of P, corresponding to this coordinate system, are given by the ordered pair (a, b).

Informally stated, vectors such as $\mathbf{i}$ and $\mathbf{j}$ that specify a coordinate system are called "basis vectors" for that system. Although in the preceding example our basis vectors were chosen to be of unit length and mutually perpendicular, this is not essential. Different basis vectors do however change the coordinates of a point, as the following example demonstrates.

Example Let $S = \{\mathbf{i}, \mathbf{j}\}$, $U = \{\mathbf{i}, 2\mathbf{j}\}$ and $V = \{\mathbf{i} + \mathbf{j}, \mathbf{j}\}$. Let P be the point $\mathbf{i} + 2\mathbf{j}$. The coordinates of P relative to each set of vectors is:

$$S \rightarrow (1, 2)$$

$$U \rightarrow (1, 1)$$

$$T \rightarrow (1, 1)$$

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The following definition makes the preceding ideas more precise and extends the notion of a coordinate system to general vector spaces.

Definition

- If V is any vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a set of vectors in V, then S is called a **basis** for V if the following two conditions hold:

  (a) S is linearly independent

  (b) S spans V

A basis is the vector space generalization of a coordinate system in 2-space and 3-space.

Theorem 3.1. *If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$ in exactly one way.*

Proof. Since S spans V, every vector in V is expressible as a linear combination of the vectors in S. To see that this expression is unique, suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

and also as

$$\mathbf{v} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n}$$

Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1 - k_1)\mathbf{v_1} + (c_2 - k_2)\mathbf{v_2} + \cdots + (c_n - k_n)\mathbf{v_n}$$

Since the right-hand side of this equation is a linear combination of vectors in S, the linear independence of S implies that

$$(c_1 - k_1) = 0, (c_2 - k_2) = 0, \ldots, (c_n - k_n)$$

That is

$$c_1 = k_1, c_2 = k_2, \ldots, c_n = k_n$$

Thus the two expressions for $\mathbf{v}$ are the same. $\qquad\square$

Definitions

- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

is the (unique) expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in $\Re^n$ constructed from these coordinates is called the **coordinate vector of** v **relative to** S; it is denoted by

$$[\mathbf{v}]_S = (c_1, c_2, \ldots, c_n)$$

- If $V = \Re^n$ and $\mathbf{v} = [\mathbf{v}]_S$ then S is called the **standard basis**.

Remark It should be noted that coordinate vectors depend not only on the basis, but also on the order in which the basis vectors are written; a change in the order of the basis vectors results in a corresponding change of order for the entries in the coordinate vectors.

Examples

1. In Example 3 of Section 2.5 it was shown that if

$$\mathbf{i} = (1, 0, 0), \;\; \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$

then $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set in $\Re^3$. This set also spans $\Re^3$ since any vector $\mathbf{v} = (a, b, c)$ can be written as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 1, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Thus, S is a basis for $\Re^3$. It is in fact a **standard basis** for $\Re^3$. The coordinates of $\mathbf{v}$ relative to the standard basis are a, b and c, so

$$[\mathbf{v}]_S = (a, b, c) = \mathbf{v}.$$

3.2 Dimension of General Vector Spaces

Definition

- A nonzero vector space V is called **finite-dimensional** if it contains a finite set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ that forms a basis. If no such set exists, V is called **infinite-dimensional**. In addition, the zero vector space is regarded as finite-dimensional.

Examples

- The vector spaces $\Re^n$ and P_n are both finite-dimensional.

- The vector space of all real valued functions defined on $(-\infty, \infty)$ is infinite-dimensional.

Theorem 3.2. *If V is finite-dimensional vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is any basis, then:*

(a) Every set with more than n vectors is linearly dependent.

(b) No set with fewer than n vectors spans V.

Proof. (a) Let $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}\}$ be any set of m vectors in V, where $m > n$. We must show that S' is linearly dependent. Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for V, each $\mathbf{w_i}, i = 1, \ldots, m$ can be expressed as a linear combination of the vectors in S, say:

$$\mathbf{w_1} = a_{11}\mathbf{v_1} + a_{12}\mathbf{v_2} + \cdots + a_{1n}\mathbf{v_n}$$

$$\mathbf{w_2} = a_{21}\mathbf{v_1} + a_{22}\mathbf{v_2} + \cdots + a_{2n}\mathbf{v_n}$$

$$\vdots$$

$$\mathbf{w_m} = a_{m1}\mathbf{v_1} + a_{m2}\mathbf{v_2} + \cdots + a_{mn}\mathbf{v_n}$$

To show that S' is linearly dependent, scalars $k_1, k_2, \ldots, k_m$ must be found, not all zero, such that

$$k_1\mathbf{w_1} + k_2\mathbf{w_2} + \cdots + k_m\mathbf{w_m} = \mathbf{0}$$

Combining the above 2 systems of equations gives

$$(k_1a_{11} + k_2a_{21} + \cdots + k_ma_{m1})\mathbf{v_1}$$
$$+ (k_1a_{12} + k_2a_{22} + \cdots + k_ma_{m2})\mathbf{v_2}$$
$$\ddots$$
$$+ (k_1a_{1n} + k_2a_{2n} + \cdots + k_ma_{mn})\mathbf{v_n} = \mathbf{0}$$

Thus, from the linear independence of S, the problem of proving that S' is a linearly dependent set reduces to showing there are scalars $k_1, k_2, \ldots, k_m$, not all zero, that satisfy

$$a_{11}k_1 + a_{21}k_2 + \cdots + a_{m1}k_m = 0$$

$$a_{12}k_1 + a_{22}k_2 + \cdots + a_{m2}k_m = 0$$

$$\vdots$$

$$a_{1n}k_1 + a_{2n}k_2 + \cdots + a_{mn}k_m = 0$$

As the system is homogenous and there are more unknowns than equations $(m > n)$ Theorem 1.2 guarantees that there are nontrivial solutions.

(b) See the handwritten notes at the end of the section.

$\square$

The last theorem essentially states this. Given a vector space V, let S be a set of n vectors which forms a basis for V, and let S' be another set of m vectors from V. If m is greater than n, S' cannot form a basis for V as the vectors in S' cannot be linearly independent. If m is less than n, S' cannot form a basis for V because it does not span V. Thus, Theorem 3.2 leads directly into one of the most important theorems in linear algebra.

Theorem 3.3. *Any two bases for a finite-dimensional vector space have the same number of vectors.*

Definition

- The **dimension** of a finite-dimensional vector space V, denoted by $dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the zero vector space has dimension zero.

Examples

1. The dimensions of some common vector spaces are given below:
$$dim(\Re^n) = n$$
$$dim(P_n) = n + 1$$
$$dim(M_{mn}) = mn$$

2. Determine a basis for and the dimension of the solution space of the homogenous system:

$$
\begin{array}{rrrrrl}
2x_1+ & 2x_2- & x_3 & & + & x_5 = & 0 \\
-x_1- & x_2+ & 2x_3- & 3x_4+ & x_5 = & 0 \\
x_1+ & x_2- & 2x_3 & & - & x_5 = & 0 \\
& & x_3+ & x_4+ & x_5 = & 0
\end{array}
$$

It was shown in Section 1.5 that the general solution is

$$x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$$

Therefore the solution vectors can be written as:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} -s - t \\ s \\ -t \\ 0 \\ t \end{bmatrix}
=
\begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix}
= s
\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ t
\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}
$$

which shows that the vectors

$$
\mathbf{v_1} =
\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\quad and \quad
\mathbf{v_2} =
\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}
$$

span the solution space. Since they are also linearly independent (see Theorem 2.7 (b)), $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis, and the solution space has dimension two.

3.3 Related Theorems

The remaining part of this section states theorems which illustrate the subtle inter-relationships between the concepts of spanning sets, linear independence, basis and

dimension. These theorems form the building blocks of many further results in linear algebra.

Theorem 3.4. Plus/Minus Theorem. *Let S be a nonempty set of vectors in a vector space V.*

(a) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V not in $span(S)$, then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.*

(b) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, then S and $S \setminus \{\mathbf{v}\}$ span the same space: that is,*

$$span(S) = span(S \setminus \{\mathbf{v}\})$$

Proofs will not be included (see the Problem Sheet solutions!), but the theorem can be visualised in $\Re^3$ as follows.

(a) Consider two linearly independent vectors in $\Re^3$. These two vectors span a plane. If you add a third vector to them that is not in the plane, then the three vectors are still linearly independent and they span the entire domain of $\Re^3$.

(b) Consider three non-collinear vectors in a plane that form a set S. The set S spans the plane. If any one of the vectors is removed from S the resulting set still spans the plane.

Theorem 3.5. *If V is an n-dimensional vector space and if S is a set in V containing exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.*

Proof. Assume that S contains exactly n vectors and spans V. To prove that S is a basis, it must be shown that S is a linearly independent set. But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors. If this vector is removed from S, then it follows from the Theorem 3.4(b) that the remaining set of $n-1$ vectors still spans V. But this is a contradiction, since it follows from Theorem 3.2(b), that no set with fewer than n vectors can span an n-dimensional vector space. Thus, S is linearly independent.

Assume that S contains exactly n vectors and is a linearly independent set. To prove that S is a basis, it must be shown that S spans V. But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in $span(S)$. If $S' = S \cup \{\mathbf{v}\}$, then Theorem

3.4(a) gives that this set of $n + 1$ vectors is still linearly independent. But this is a contradiction, since it follows from Theorem 3.2(a) that in an n-dimensional vector space, no set containing more than n vectors can be linearly independent. Thus S spans V. $\square$

Theorem 3.6. *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(a) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(b) *If S is a linearly independent set that is not a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

Proof. (a) The proof is constructive and is called the **left to right algorithm**.

Let $\mathbf{v_{c_1}}$ be the first nonzero vector in the set S. Choose the next vector in the list which is not a linear combination of $\mathbf{v_{c_1}}$ and call it $\mathbf{v_{c_2}}$. Find the next vector in the list which is not a linear combination of $\mathbf{v_{c_1}}$ and $\mathbf{v_{c_2}}$ and call it $\mathbf{v_{c_3}}$. Continue in such a way until the number of vectors chosen equals $dim(V)$.

(b) This proof is also constructive.

Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ be a linearly independent set in V and let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis for V. To extend the basis, simply apply the left to right algorithm to the set

$$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

(Note that this set spans V because it contains a basis within it.) This will select a basis for V that commences with $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$

$\square$

Theorem 3.7. *If W is a subspace of a finite-dimensional vector space V, then $dim(W) \leq dim(V)$; moreover, if $dim(W) = dim(V)$, then $W = V$*

Proof. Let $S = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_r}\}$ be a basis for W. Either S is also a basis for V or it is not. If it is, then $dim(W) = dim(V) = r$ and $W = V$. If it is not, then by the previous theorem, vectors can be added to the linearly independent set S to make it into a basis for V, so $dim(W) < dim(V)$. Thus, $dim(W) \leq dim(V)$ in all cases. $\square$

3.4   Change of Basis

¿From the previous section, we know that the basis of a vector space is the set of vectors that specify the coordinate system. A vector space may have an infinite number of bases, but each basis contains the same number of vectors. The number of vectors in the basis is called the dimension of the vector space.

The coordinate vector of a point changes with any change in the basis used. If the basis for a vector space is changed from some old basis β to some new basis γ, how is the old coordinate vector $[\mathbf{v}]_\beta$ of a vector $\mathbf{v}$ related to the new coordinate vector $[\mathbf{v}]_\gamma$? The following theorem answers that question.

Theorem 3.8. *If the basis for a vector space is changed from some old basis* $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ *to some new basis* $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, *then the old coordinate vector* $[\mathbf{w}]_\beta$ *is related to the new coordinate vector* $[\mathbf{w}]_\gamma$ *of the same vector* $\mathbf{w}$ *by the equation*

$$[\mathbf{w}]_\gamma = P[\mathbf{w}]_\beta$$

where the columns of P *are the coordinate vectors of the old basis vectors relative to the new basis; that is, the column vectors of* P *are*

$$[\mathbf{u_1}]_\gamma, [\mathbf{u_2}]_\gamma, \ldots, [\mathbf{u_n}]_\gamma$$

P is called the **change of basis matrix** *or the* **change of coordinate matrix**.

Proof. Let V be a vector space with a basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and a new basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. Let $\mathbf{w} \in V$. Therefore $\mathbf{w}$ can be expressed as:

$$\mathbf{w} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_n\mathbf{u_n}$$

Thus we have

$$[\mathbf{w}]_\beta = (a_1, a_2, \ldots, a_n)$$

As γ is also a basis of V the elements of β can be expressed as follows

$$\mathbf{u_1} = p_{11}\mathbf{v_1} + p_{12}\mathbf{v_2} + \cdots + p_{1n}\mathbf{v_n}$$

$$\mathbf{u_2} = p_{21}\mathbf{v_1} + p_{22}\mathbf{v_2} + \cdots + p_{2n}\mathbf{v_n}$$

$$\vdots$$

$$\mathbf{u_n} = p_{n1}\mathbf{v_1} + p_{n2}\mathbf{v_2} + \cdots + p_{nn}\mathbf{v_n}$$

Combining this system of equations with the above expression for $\mathbf{w}$ gives

$$\mathbf{w} = (p_{11}a_1 + p_{21}a_2 + \cdots + p_{n1}a_n)\mathbf{v_1}$$

$$+(p_{12}a_1 + p_{22}a_2 + \cdots + p_{n2}a_n)\mathbf{v_2}+$$

$$\vdots$$

$$+(p_{1n}a_1 + p_{2n}a_2 + \cdots + p_{nn}a_n)\mathbf{v_n}$$

and thus it can be seen that

$$[\mathbf{w}]_\gamma = \begin{bmatrix} p_{11}a_1 + p_{21}a_2 + \cdots + p_{n1}a_n \\ p_{12}a_1 + p_{22}a_2 + \cdots + p_{n2}a_n \\ \vdots \\ p_{1n}a_1 + p_{2n}a_2 + \cdots + p_{nn}a_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Thus

$$[\mathbf{w}]_\gamma = P[\mathbf{w}]_\beta$$

where P's columns are

$$[\mathbf{u_1}]_\gamma, [\mathbf{u_2}]_\gamma, \ldots, [\mathbf{u_n}]_\gamma$$

$\square$

Example

1. Consider the bases $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ for $\Re^2$, where

$$\mathbf{v_1} = (1,0); \quad \mathbf{v_2} = (0,1); \quad \mathbf{u_1} = (1,1); \quad \mathbf{u_2} = (2,1)$$

(a) Find the change of basis matrix from β to γ. First we must find the coordinate vectors of the old basis vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ relative to the new basis β. By inspection:

$$\mathbf{u_1} = \mathbf{v_1} + \mathbf{v_2}$$

$$\mathbf{u_2} = 2\mathbf{v_1} + \mathbf{v_2}$$

so that

$$[\mathbf{u_1}]_\gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad and \quad [\mathbf{u_2}]_\gamma = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus the change of basis matrix from β to γ is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) Use the change of basis matrix to find $[\mathbf{v}]_\gamma$ if

$$[\mathbf{v}]_\beta = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

¿From Theorem 3.8

$$[\mathbf{v}]_\gamma = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

A quick check on your answer is to recover the vector $\mathbf{v}$ from both $[\mathbf{v}]_\gamma$ and $[\mathbf{v}]_\beta$. In this example $\mathbf{v} = -3\mathbf{u_1} + 5\mathbf{u_2} = 7\mathbf{v_1} + 2\mathbf{v_2} = (7, 2)$.

4 ORTHONORMAL BASES

In many problems involving vector spaces, the problem solver is free to choose any basis for the vector space that seems appropriate. In $\Re^n$ the solution of a problem is often greatly simplified by choosing a basis in which the vectors are orthogonal to one another. In this section it shall be shown how such a basis can be obtained.

Definitions

- Let $\mathbf{u}, \mathbf{v} \in \Re^n$. Then the operation defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

  where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

  maps $\Re^n \to \Re$ and is referred to as the **Euclidean inner product** or the **dot product**.

- Two vectors $\mathbf{u}, \mathbf{v} \in \Re^n$ are said to be **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors and both $\mathbf{u}$ and $\mathbf{v}$ have a magnitude of one, then $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthonormal**.

- A set of vectors in $\Re^n$, on which the dot product is defined, with the property that all pairs of distinct vectors in the set are orthogonal is called an **orthogonal set**. An orthogonal set in which each vector has a magnitude of one is called an **orthonormal set**.

The proof of the following result is important, since it provides an algorithm for converting an arbitrary basis into an orthonormal basis.

Theorem 4.1. *Every subspace of the vector space $\Re^n$ has an orthonormal basis.*

Proof. Let V be any nonzero subspace of $\Re^n$, and let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be any basis for V. It suffices to show that V has an orthogonal basis, since the vectors in the orthogonal basis can be normalized to produce an orthonormal basis for V. The following sequence of steps will produce an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ for V.

Step 1 Let $\mathbf{v_1} = \mathbf{u_1}$.

Step 2 Obtain a vector $\mathbf{v_2}$ that is orthogonal to $\mathbf{v_1}$ by computing the component of $\mathbf{u_2}$ that is orthogonal to the space W_1 spanned by $\mathbf{v_1}$. This can be done using the formula:

$$\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1}$$

Of course, if $\mathbf{v_2} = \mathbf{0}$, then $\mathbf{v_2}$ is not a basis vector. But this cannot happen, since it would then follow from the preceding formula for $\mathbf{v_2}$ that

$$\mathbf{u_2} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \right) \mathbf{u_1}$$

which says that $\mathbf{u_2}$ is a multiple of $\mathbf{u_1}$, contradicting the linear independence of the basis $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$.

Step 3 To construct a vector $\mathbf{v_3}$ that is orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$, compute the component of $\mathbf{u_3}$ orthogonal to the space W_2 spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ using the formula:

$$\mathbf{v_3} = \mathbf{u_3} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \right) \mathbf{v_2}$$

As in step 2, the linear independence of $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ ensures that $\mathbf{v_3} \neq \mathbf{0}$.

Step 4 To determine a vector $\mathbf{v_4}$ that is orthogonal to $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$, compute the component of $\mathbf{u_4}$ orthogonal to the space W_3 spanned by $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ using the formula

$$\mathbf{v_4} = \mathbf{u_4} - \left(\frac{\mathbf{u_4} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} - \left(\frac{\mathbf{u_4} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \right) \mathbf{v_2} - \left(\frac{\mathbf{u_4} \cdot \mathbf{v_3}}{\mathbf{v_3} \cdot \mathbf{v_3}} \right) \mathbf{v_3}$$

Continuing in this way, an orthogonal set of vectors, $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$, will be obtained after m steps. Since V is an m-dimensional vector space and every orthogonal set is linearly independent, the set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is an orthogonal basis for V. □

This preceding step-by-step construction for converting an arbitrary basis into an orthogonal basis is called the **Gramm-Schmidt process**.

Example: *THE GRAMM-SCHMIDT PROCESS*

1. Consider the vector space $\Re^3$ with the Euclidean inner product. Apply the Gramm-Schmidt process to transform the basis vectors $\mathbf{u_1} = (1, 1, 1), \mathbf{u_2} = (0, 1, 1), \mathbf{u_3} = (0, 0, 1)$ into an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$.

Step 1

$$\mathbf{v_1} = \mathbf{u_1} = (1, 1, 1)$$

Step 2

$$\begin{aligned}
\mathbf{v_2} &= \mathbf{u_2} - \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} \\
&= (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

Step 3

$$\begin{aligned}
\mathbf{v_3} &= \mathbf{u_3} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right) \mathbf{v_2} \\
&= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
&= \left(0, -\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}$$

Thus,

$$\mathbf{v_1} = (1, 1, 1), \quad \mathbf{v_2} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v_3} = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

form an orthogonal basis for $\Re^3$. The norms of these vectors are

$$\|\mathbf{v_1}\| = \sqrt{3}, \quad \|\mathbf{v_2}\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v_3}\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for $\Re^3$ is

$$\mathbf{q_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{q_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\mathbf{q_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

The Gramm-Schmidt process with subsequent normalization not only converts an arbitrary basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ into an orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\}$, but it does it in such a way that for $k \geq 2$ the following relationships hold:

- $\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_k}\}$ is an orthonormal basis for the space spanned by $\{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$.

- $\mathbf{q_k}$ is orthogonal to $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{k-1}}\}$.

5 ROW, COLUMN AND NULL SPACE

This section is devoted to the study of these three important vector spaces associated with a matrix. Such study leads to a deeper understanding of the relationship between the solutions of a linear system and the properties of its coefficient matrix.

5.1 Basic Definitions

Definitions

- For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r_1} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r_2} &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\ \ \vdots \qquad\qquad \vdots \\ \mathbf{r_m} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in $\Re^n$ formed from the rows of A are called the **row vectors** of A.

- The vectors

$$\mathbf{c_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \mathbf{c_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\Re^m$ formed from the columns of A are called the **column vectors** of A.

- The subspace of $\Re^n$ spanned by the row vectors of A is called the **row space** of A. The row space of A is denoted by $R(A)$.

$$R(A) = span\{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}\}$$

- The subspace of $\Re^m$ spanned by the column vectors of A is called the **column space** of A. The column space of A is denoted by $C(A)$.

$$C(A) = span\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}\}$$

- The solution space of the homogenous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\Re^n$, is called the **null space** of A. The null space of A is denoted by $N(A)$.

$$N(A) = \{\mathbf{x} \in \Re^n | A\mathbf{x} = \mathbf{0}\}$$

We now verify the claim that $N(A)$ is a subspace of $\Re^n$: Clearly $N(A) \subseteq \Re^n$. Let $\mathbf{u}$ and $\mathbf{v} \in N(A)$. Then

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}.$$

Since matrix multiplication distributes over matrix addition,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and we have that $\mathbf{u} + \mathbf{v} \in N(A)$. Also if k is any scalar,

$$A(k\mathbf{u}) = kA\mathbf{u} = k\mathbf{0} = \mathbf{0}$$

and we have that $k\mathbf{u} \in N(A)$. Hence the null space is closed under addition and scalar multiplication and is therefore, by Theorem 2.3, a subspace of $\Re^n$.

Example Let A be the 2×3 matrix shown below

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 9 & 2 \end{bmatrix}$$

Then the row vectors are

$$\mathbf{r_1} = [1, 3, 2] \quad \text{and} \quad \mathbf{r_2} = [4, 9, 2]$$

$$R(A) = span\{\mathbf{r_1}, \mathbf{r_2}\}$$

and the column vectors are

$$\mathbf{c_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \quad \text{and} \quad \mathbf{c_3} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$C(A) = span\{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}\}$$

Finding the null space of A amounts to finding the solution space of the homogenous system of linear equations $A\mathbf{x} = \mathbf{0}$. The augmented matrix $[A \mid \mathbf{0}]$ is

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 9 & 2 & 0 \end{bmatrix}$$

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The reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Since there is no leading one in column three corresponding to the variable x_3, x_3 is the independent variable. Let $x_3 = t$ for an arbitrary scalar t. Then the general solution is

$$x_2 = -2t \quad and \quad x_1 = 4t$$

So any solution vector x is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Therefore $N(A)$ is spanned by the vector $(4, 2, 1)^T$, and this is also a basis for $N(A)$.

However, as usual, we would like to express these subspaces in terms of bases rather than just spanning sets. Finding a basis for $N(A)$ is equivalent to finding a basis for the solution space of the homogenous system $AX = 0$, and this procedure has been demonstrated previously (as in the previous example, Example 2 on page 33, or Questions 5 and 6 from Problem Sheet Three). The next section looks at an algorithm used for determining bases for $R(A)$ and $C(A)$.

5.2   Finding Bases for R(A) and C(A)

We first state a few theorems regarding the effect of elementary row operations on the subspaces $R(A)$ and $C(A)$.

Performing an elementary row operation on an augmented matrix does not change the solution set of the corresponding linear system. In particular, applying an elementary row operation to a matrix A does not change the solution set of the corresponding linear system $A\mathbf{x} = \mathbf{0}$, or, stated another way, it does not change the null space of A. Thus we have:

Theorem 5.1. *Elementary row operations do not change the null space of a matrix.*

The following theorem is a companion to Theorem 5.1.

Theorem 5.2. *Elementary row operations do not change the row space of a matrix.*

Proof. Suppose that the row vectors of a matrix A are $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}$ and let B be obtained from A by performing an elementary row operation. It shall be shown that $R(A) = R(B)$.

Consider the possibilities:

Case 1: If the row operation is a row interchange $(\mathbf{r_i} \leftrightarrow \mathbf{r_j})$, then B and A have the same row vectors and consequently have the same row space.

Case 2: If the row operation is multiplication of a row by a nonzero scalar t $(\mathbf{r_i} \rightarrow t\mathbf{r_i})$ or the addition of a multiple of one row to another $(\mathbf{r_i} \rightarrow \mathbf{r_i} + t\mathbf{r_j})$, then the row vectors $\mathbf{r'_1}, \mathbf{r'_2}, \ldots, \mathbf{r'_m}$ of B are linear combinations of the row vectors $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}$ of A; thus, they lie in the row space of A. Since a subspace is closed under addition and scalar multiplication, all linear combinations of $\mathbf{r'_1}, \mathbf{r'_2}, \ldots, \mathbf{r'_m}$ will also lie in the row space of A. Thus $R(B) \subseteq R(A)$.

Now observe that since B is obtained from A by performing an elementary row operation, A can be obtained from B by performing the inverse of that operation. Then the argument above, when reversed, shows that $R(A) \subseteq R(B)$ and hence that $R(A) = R(B)$. $\qquad\square$

In light of the previous two theorems, one might be forgiven for thinking that elementary row operations have no effect on the *column* space of a matrix. In fact, this is definitely not the case. Elementary row operations on a matrix do change the column space. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

The second column is a scalar multiple of the first, so the column space of A consists of all scalar multiples of the first column vector. However, if we perform the elementary row operation $\mathbf{r_2} \rightarrow \mathbf{r_2} + -2\mathbf{r_1}$, we obtain

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Here again the second column of B is a scalar multiple of the first, so the column space of B consists of all scalar multiples of the first column vector. However, $span([2 \ 4]^T) \neq span([2 \ 0]^T)$; that is, $C(A) \neq C(B)$.

Although elementary row operations can change the column space of a matrix A, the relationships of linear independence or linear dependence that exist between the column vectors of A prior to a row operation, also exist between the column

vectors of the resulting matrix. To show this, let B be the matrix that results by performing an elementary row operation on an $m \times n$ matrix A. By Theorem 5.1, the two homogenous linear systems

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set. Thus, the first system has a nontrivial solution if and only if the same is true of the second. This implies that the column vectors of A are linearly independent if and only if the same is true of B. This conclusion can also be applied to any subset of the column vectors (check it!). Thus, we have the following result.

Theorem 5.3. *If A and B are row-equivalent matrices, then:*

(a) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(b) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

Theorems 5.1, 5.2 and 5.3 lead to the following theorem, which describes bases for the row and column spaces of a matrix in row-echelon form.

Theorem 5.4. *If a matrix R is in row-echelon form, then the row vectors containing the leading ones (that is, the non-zero row vectors) form a basis for the row space of R, and the column vectors containing those leading ones form a basis for the column space of R.*

Examples

1. The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form. From Theorem 5.4 the vectors

$$
\begin{aligned}
\mathbf{r_1} &= \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix} \\
\mathbf{r_2} &= \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix} \\
\mathbf{r_3} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

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form a basis for the row space of R, and the vectors

$$\mathbf{c_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

2. Find bases for the row and column space of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Since performing elementary row operations on a matrix does not change its row space, a basis for the row space of any row-echelon form of A is a basis for the row space of A. One row-echelon form of A is

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 5.4 the nonzero row vectors of R form a basis for the row space of R, and hence form a basis for the row space of A. These basis vectors are

$$\begin{aligned} \mathbf{r_{1_R}} &= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix} \\ \mathbf{r_{2_R}} &= \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix} \\ \mathbf{r_{3_R}} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

Keeping in mind that A and R may have different column spaces, it is not possible to find a basis for the column space of A *directly* from the column vectors of R. However, it follows from Theorem 5.4 that if a set of column vectors of R form a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A. The first, third, and fifth columns of R contain the leading ones, so

$$\mathbf{c_{1_R}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_{3_R}} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_{5_R}} = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R; thus the first, third and fifth column vectors of A, namely,

$$c_{1_A} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad c_{3_A} = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad c_{5_A} = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

3. Find a basis for the space spanned by the vectors

$$\mathbf{v_1} = (1, -2, 0, 0, 3), \quad \mathbf{v_2} = (2, -5, -3, -2, 6), \quad \mathbf{v_3} = (0, 5, 15, 10, 0)$$

$$\mathbf{v_4} = (2, 6, 18, 8, 6)$$

If we define the matrix A to be the matrix whose rows are precisely these row vectors, then the space spanned by these vectors is just the row space of A. Thus a basis for $R(A)$ is a basis for the space spanned by these vectors. (We note that this basis will not necessarily contain any of the original vectors, unlike the basis obtained by performing the left-to-right algorithm.)

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

The matrix is reduced to row-echelon form:

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors in this matrix are

$$\mathbf{w_1} = (1, -2, 0, 0, 3), \quad \mathbf{w_2} = (0, 1, 3, 2, 0), \quad \mathbf{w_3} = (0, 0, 1, 1, 0).$$

These vectors form a basis for the row space and consequently form a basis for the subspace of $\Re^5$ spanned by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and $\mathbf{v_4}$.

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Note that in these examples a basis for the column space of A consisted of column vectors of A, while a basis for the row space of A consisted of row vectors of a row-echelon form of A. The following example demonstrates how to find a basis for the space spanned by a set S of vectors, which consists entirely of vectors in S. We shall define a matrix A whose *columns* are the vectors in S, and then find a basis for $C(A)$. (This procedure is computationally faster than the left-to-right algorithm.)

4. (a) Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ where

$$\mathbf{v_1} = (1, -2, 0, 3), \quad \mathbf{v_2} = (2, -5, -3, 6)$$

$$\mathbf{v_3} = (0, 1, 3, 0), \quad \mathbf{v_4} = (2, -1, 4, -7), \quad \mathbf{v_5} = (5, -8, 1, 2)$$

Find a subset of S that forms a basis for the space spanned by S.

Begin by constructing a matrix A that has $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_5}$ as its column vectors.

$$A = [\mathbf{v_1} \mid \mathbf{v_2} \mid \mathbf{v_3} \mid \mathbf{v_4} \mid \mathbf{v_5}] = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

The problem is easily solved by finding a basis for $C(A)$. For reasons that will become evident in part (b) the matrix is reduced to *reduced* row-echelon form. Denoting the column vectors of the resulting matrix by $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ and $\mathbf{w_5}$ yields

$$B = rref(A) = [\mathbf{w_1} \mid \mathbf{w_2} \mid \mathbf{w_3} \mid \mathbf{w_4} \mid \mathbf{w_5}] = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's occur in columns 1, 2 and 4, so by Theorem 5.4

$$\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_4}$$

is a basis for the column space of B and consequently

$$\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}$$

form a basis for the column space of the A and hence for the space spanned by S.

(b) Express the vectors in S that are not in the basis as a linear combination of the basis vectors.

Here the reason for converting the matrix A to reduced row-echelon form in part (a) becomes clear. When the matrix is in reduced row-echelon form, it is much easier to see the linear dependence of $\mathbf{w_3}$ and $\mathbf{w_5}$ on the vectors in the basis. By inspection, the linear combinations are

$$\mathbf{w_3} = 2\mathbf{w_1} - \mathbf{w_2}$$
$$\mathbf{w_5} = \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_4}$$

We call these the **dependency equations**. The corresponding relationships in the original matrix are:

$$\mathbf{v_3} = 2\mathbf{v_1} - \mathbf{v_2}$$
$$\mathbf{v_5} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_4}$$

5.2.1 Extending a Linearly Independent Set to a Basis

It was stated in Theorem 3.6(b) that if V is a vector space of dimension n and if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ form an linearly independent set in V such that $r < n$ then there exists vectors $\mathbf{v_{r+1}}, \mathbf{v_{r+2}}, \ldots, \mathbf{v_n}$ such that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ form a basis for V. The next example demonstrates how to extend an linearly independent family to a basis for V.

Example

1. Take $V = \Re^4$, and also let $\mathbf{v_1}$ and $\mathbf{v_2}$ be

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ 6 \\ -8 \\ 100 \end{bmatrix}$$

It is clear that $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. As specified in the proof of Theorem 3.6(b), let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$, where $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ is the standard basis for V. Let A be the matrix

$$A = [\mathbf{v_1} \mid \mathbf{v_2} \mid \mathbf{u_1} \mid \mathbf{u_2} \mid \mathbf{u_3} \mid \mathbf{u_4}]$$

A basis for the column space of A is a basis for V, and in particular, this basis will contain the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Note that S spans V since it contains a

basis for V. Finding the rref of A:

$$B = rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{u_1}$ and $\mathbf{u_2}$ form a basis for $\Re^4$ and we have extended the linearly independent set to a basis.

5.3   Rank and Nullity

Theorem 5.5. *If A is any matrix, then the row space and column space of A have the same dimension.*

Proof. Let R be the reduced row-echelon form of A. It follows from Theorem 5.2 that

$$dim(\text{row space of } A) = dim(\text{row space of } R)$$

and it follows from Theorem 5.3 that

$$dim(\text{column space of } A) = dim(\text{column space of } R).$$

Thus, the proof would be complete if it could be shown that the row space and column space of the matrix R have the same dimension. The dimension of the row space of R is the number of nonzero rows and the dimension of the column space of R is the number of columns that contain leading ones (Theorem 5.4). However, the nonzero rows are precisely the rows in which the leading ones occur, so the number of leading ones and the number of nonzero rows is the same. Thus the row space and column space of R have the same dimension and $dim(R(A)) = dim(C(A))$. $\square$

Definitions

- The (common) dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by $rank(A)$.

- The dimension of the null space of A is called the **nullity** of A and is denoted by $nullity(A)$.

The following theorem establishes an important relationship between the rank and nullity of a matrix.

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Theorem 5.6. Rank Plus Nullity Theorem *If A is a matrix with n columns, then*

$$rank(A) + nullity(A) = n$$

Proof. Since A has n columns, the homogenous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two categories: the leading variables and the free variables. Thus,

$$\begin{bmatrix} number\ of\ leading \\ variables \end{bmatrix} + \begin{bmatrix} number\ of\ free \\ variables \end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading ones in the reduced row-echelon form of A, and this is the rank of A. Thus,

$$rank(A) + \begin{bmatrix} number\ of\ free \\ variables \end{bmatrix} = n$$

The number of free variables is equal to nullity of A. This is so because the nullity of A is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the number of parameters in the general solution, which is the same as the number of free variables. Thus,

$$\text{rank}(A) + \text{nullity}(A) = n$$

$\square$

5.4   Related Theorems

The following theorem provides conditions under which a linear system of m equations in n unknowns is guaranteed to be consistent.

Theorem 5.7. (The Consistency Theorem). *If $A\mathbf{x} = \mathbf{0}$ is a linear system of m equations in n unknowns, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{b}$ is consistent.

(b) $\mathbf{b}$ is in the column space of A.

(c) The coefficient matrix A and the augmented matrix $[A \mid \mathbf{b}]$ have the same rank.

Proof. It suffices to prove the two equivalencies (a)⇔(b) and (b)⇔(c), since it will then follow from logic that (a)⇔(c).

(a)$\Leftrightarrow$(b) Consider the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Let $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ denote the column vectors of A. Then the product $A\mathbf{x}$ can be expressed as

$$A\mathbf{x} = x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n}$$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of m equations in n unknowns can be written as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{b}$$

from which it can be concluded that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is expressible as a linear combination of the column vectors of A or, equivalently, if and only if $\mathbf{b}$ is in the column space of A.

(b)$\Rightarrow$(c) It shall be shown that if $\mathbf{b}$ is in the column space of A, then the column spaces of A and $[A \mid \mathbf{b}]$ are actually the same, from which it follows that the two matrices have the same rank.

By definition, the column space of a matrix is the space spanned by its column vectors, so the column spaces of A and $[A \mid \mathbf{b}]$ can be expressed as

$$span\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}\} \text{ and } span\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$$

respectively. If $\mathbf{b}$ is in the column space of A, then each vector in the set $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$ is a linear combination of the vectors in $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}\}$ and conversely. Thus, from Theorem 2.5 the column spaces of A and $[A \mid \mathbf{b}]$ are the same.

(c)$\Rightarrow$(b) Assume that A and $[A \mid \mathbf{b}]$ have the same rank r. By Theorem 3.6(a), there is some subset of the column vectors of A that forms a basis for the column space of A. Suppose that those column vectors are

$$\mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_r}$$

These r linearly independent vectors also belong to the r-dimensional column space of $[A \mid \mathbf{b}]$, thus they form a basis for the column space of $[A \mid \mathbf{b}]$. This means that $\mathbf{b}$ is expressible as a linear combination of $\mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_r}$, and consequently $\mathbf{b}$ lies in the column space of A.

$\square$

Example This example demonstrates that condition (a) of Theorem 5.8 holds if and only if condition (c) holds. It helps to think of the rank of a matrix as the number of nonzero rows in its reduced row-echelon form. The augmented matrix for the system

$$
\begin{array}{rrrrr}
x_1- & 2x_2- & 3x_3+ & 2x_4 = & -4 \\
-3x_1+ & 7x_2- & x_3+ & x_4 = & -3 \\
2x_1- & 5x_2+ & 4x_3- & 3x_4 = & 7 \\
-3x_1+ & 6x_2+ & 9x_3- & 6x_4 = & -1
\end{array}
$$

is

$$
\begin{bmatrix}
1 & -2 & -3 & 2 & -4 \\
-3 & 7 & -1 & 1 & -3 \\
2 & -5 & 4 & -3 & 7 \\
-3 & 6 & 9 & -6 & -1
\end{bmatrix}
$$

which has the following reduced row-echelon form:

$$
\begin{bmatrix}
1 & 0 & -23 & 16 & 0 \\
0 & 1 & -10 & 7 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Because of the third row the system is inconsistent. However, it is also because of this row that the reduced row-echelon form of the augmented matrix has fewer zero rows than the reduced row-echelon form of the coefficient matrix. This forces the coefficient matrix and the augmented matrix for the system to have different ranks.

The Consistency Theorem was concerned with the conditions under which a linear system $A\mathbf{x} = \mathbf{b}$ is consistent for a *specific* vector $\mathbf{b}$. The following theorem is concerned with conditions under which a linear system is consistent for *all possible* choices of $\mathbf{b}$.

Theorem 5.8. *If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $m \times 1$ matrix $\mathbf{b}$.*

(b) *The column vectors of A span $\Re^m$.*

(c) $rank(A) = m.$

Proof. It suffices to prove the two equivalences (a)$\Leftrightarrow$(b) and (a)$\Leftrightarrow$(c), since it will then follow from logic that (b)$\Leftrightarrow$(c).

(a)$\Leftrightarrow$(b) As in the proof of the previous theorem, $A\mathbf{x} = \mathbf{b}$ can be expressed as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{b}$$

from which it can be concluded that $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$ if and only if every such $\mathbf{b}$ is expressible as a linear combination of the column vectors $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$, or equivalently, if and only if these column vectors span $\Re^m$.

(a)$\Rightarrow$(c) From the assumption that $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$, and from parts (a) and (b) of Theorem 5.7, it follows that every vector $\mathbf{b}$ in $\Re^m$ lies in the column space of A; that is, the column space of A is all of $\Re^m$. Thus $\text{rank}(A) = \dim(\Re^m) = m$.

(c)$\Rightarrow$(a) From the assumption that $\text{rank}(A) = m$, it follows that the column space of A is a subspace of $\Re^m$ of dimension m, and hence must be all of $\Re^m$ by Theorem 3.7. It now follows from parts (a) and (b) of Theorem 5.7 that $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\Re^m$, since every such $\mathbf{b}$ is in the column space of A.

$\square$

Example

1. A linear system with more equations than unknowns is called an **overdetermined linear system**. If $A\mathbf{x} = \mathbf{b}$ is an overdetermined linear system of m equations in n unknowns (so that $m > n$), then the column vectors of A do not span $\Re^m$. (Theorem 3.2 (b) gives that no set of fewer than m vectors will span $\Re^m$.) The previous theorem tells us that there exists at least one vector $\mathbf{b}$ in $\Re^m$ for which there are no solutions to $A\mathbf{x} = \mathbf{b}$.

Theorem 5.9. *If A is an $m \times n$ matrix, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) The column vectors of A are linearly independent.

(c) $A\mathbf{x} = \mathbf{b}$ has at most one solution (none or one) for every $m \times 1$ matrix $\mathbf{b}$.

Proof. It suffices to prove the two equivalencies (a)⇔(b) and (a)⇔(c), since it will then follow from logic that (b)⇔(c).

(a)⇔(b) If $c_1, c_2, \ldots, c_n$ are the column vectors of A, then the linear system $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = \mathbf{0}.$$

Now $c_1, c_2, \ldots, c_n$ are linearly independent if and only if this equation is satisfied only by $x_1 = x_2 = \cdots = x_n = 0$ if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(a)⇒(c) Assume that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Either $A\mathbf{x} = \mathbf{b}$ is consistent or it is not. If it is not consistent, then there are no solutions to $A\mathbf{x} = \mathbf{b}$. If $A\mathbf{x} = \mathbf{b}$ is consistent, let $\mathbf{x_0}$ and $\mathbf{y}$ be (not necessarily distinct) solutions. We wish to show that $\mathbf{x_0} = \mathbf{y}$. Now

$$A(\mathbf{x_0} - \mathbf{y}) = A\mathbf{x_0} - A\mathbf{y} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x_0} - \mathbf{y} = \mathbf{0}$ and so $\mathbf{x_0} = \mathbf{y}$. Thus if $A\mathbf{x} = \mathbf{b}$ is consistent, there is a unique solution. Thus $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$.

(c)⇒(a) Assume that $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$. Then, in particular, $A\mathbf{x} = \mathbf{0}$ has at most one solution. Thus, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

□

The last few theorems have dealt with $m \times n$ matrices. This next theorem deals exclusively with square matrices, and their invertibility.

Theorem 5.10. *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The column vectors of A are linearly independent.

(d) The row vectors of A are linearly independent.

(e) The column vectors of A span $\Re^n$.

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(f) The row vectors of A span $\Re^n$.

(g) The column vectors of A form a basis for $\Re^n$.

(h) The row vectors of A form a basis for $\Re^n$.

(i) A has rank n.

(j) A has nullity 0.

Proof. We shall show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (j) \Rightarrow (a)$

(a)$\Rightarrow$(b)

Assume A is invertible and let $\mathbf{x_0}$ be any solution of $A\mathbf{x} = \mathbf{0}$; thus, $A\mathbf{x_0} = \mathbf{0}$. Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}(A\mathbf{x_0}) = A^{-1}\mathbf{0}$, or $(A^{-1}A)\mathbf{x_0} = \mathbf{0}$, or $I\mathbf{x_0} = \mathbf{0}$, or $\mathbf{x_0} = \mathbf{0}$. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b)$\Rightarrow$(c)

If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then by Theorem 5.9 the column vectors of A are linearly independent.

(c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f) $\Rightarrow$(g)$\Rightarrow$(h)

This follows from Theorems 5.4, 3.5 and the fact that $\Re^n$ is an n-dimensional vector space.

(h)$\Rightarrow$(i)

If the n row vectors of A form a basis for $\Re^n$, then the row space of A is n-dimensional and A has rank n.

(i)$\Rightarrow$(j)

This follows from the (Rank Plus Nullity) Theorem 5.6.

(j)$\Rightarrow$(a)

If A has nullity 0, then the rref of A has no zero rows. Since A is square, this means that $rref(A) = I_n$, and the rref of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $[I_n|\mathbf{b'}]$, for some vector $\mathbf{b'} = [b'_1, b'_2, \ldots, b'_n]^T$ in $\Re^n$. Thus the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $x_1 = b'_1, x_2 = b'_2, \ldots, x_n = b'_n$. But $\mathbf{b'}$ is obtained from $\mathbf{b}$ by some sequence of elementary row operations, and thus each b'_i, $1 \le i \le n$ is a linear

combination of $b_1, b_2, \ldots, b_n$. This means that each x_i is then a linear combination of $b_1, b_2, \ldots, b_n$, and we may write

$$\mathbf{x} = B\mathbf{b}$$

where B is the coefficient matrix of the system of n equations expressing each x_i as a linear combination of $b_1, b_2, \ldots, b_n$. So

$$A\mathbf{x} = A(B\mathbf{b}) = (AB)\mathbf{b} = \mathbf{b}.$$

Thus AB is the identity matrix I_n, and A is invertible. $\qquad\square$

6 LINEAR TRANSFORMATIONS

Definitions

- If $T : V \to W$ is a function that maps a vector space V into a vector space W, then T is called a **linear transformation** from V to W if for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and all scalars c

  (a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

  (b) $T(c\mathbf{u}) = cT(\mathbf{u})$

- In the special case where $V = W$, the linear transformation $T : V \to V$ is called a **linear operator** on V.

Examples

1. Let A be an $m \times n$ matrix and let $T_A : \Re^n \to \Re^m$ be the function defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \Re^n$. Let $\mathbf{u}$ and $\mathbf{v} \in \Re^n$, then

$$T_A(\lambda\mathbf{u} + \mu\mathbf{v}) = A(\lambda\mathbf{u} + \mu\mathbf{v})$$

$$= \lambda A\mathbf{u} + \mu A\mathbf{v}$$

$$= \lambda T_A(\mathbf{u}) + \mu T_A(\mathbf{v})$$

and thus T_A is a linear transformation.

2. If I is the $n \times n$ identity matrix, then for every vector $\mathbf{x}$ in $\Re^n$

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in $\Re^n$ onto itself. $T_I(\mathbf{x})$ is called the **identity operator** on $\Re^n$.

3. Let A, B and X be $n \times n$ matrices. Then $Y = AX - XB$ is also $n \times n$.

   Let $V = M_{n \times n}(\Re)$ be the vector space of all $n \times n$ matrices. Then $Y(X) = AX - XB$ defines a transformation $T : V \to V$. The transformation is linear since:

$$
\begin{aligned}
T(\lambda X_1 + \mu X_2) &= A(\lambda X_1 + \mu X_2) - (\lambda X_1 + \mu X_2)B \\
&= \lambda AX_1 + \mu AX_2 - \lambda X_1 b - \mu X_2 B \\
&= \lambda(AX_1 - X_1 B) + \mu(AX_2 - X_2 B) \\
&= \lambda T(X_1) + \mu T(X_2)
\end{aligned}
$$

Theorem 6.1. *If $T : \Re^n \to \Re^m$ is a linear transformation, then there exists an $m \times n$ matrix A such that $T = T_A$.*

Proof. Let $\mathbf{x} \in \Re^n$ and let $T : \Re^n \to \Re^m$. Also let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ be the standard basis for $\Re^n$. We may write $\mathbf{x}$ as

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{e_1} + x_2 \mathbf{e_2} + \cdots + x_n \mathbf{e_n}
$$

Then

$$
\begin{aligned}
T(\mathbf{x}) &= x_1 T(\mathbf{e_1}) + x_2 T(\mathbf{e_2}) + \cdots + x_n T(\mathbf{e_n}) \\
&= [T(\mathbf{e_1}) \,|\, T(\mathbf{e_2}) \,|\, \cdots \,|\, T(\mathbf{e_n})]\mathbf{x} \\
&= A\mathbf{x}
\end{aligned}
$$

where A is the matrix whose columns are the transformation of $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$. Therefore $T = T_A$. $\square$

Example

1. Find the 2×2 matrix A such that $T = T_A$ has the property that

$$
T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

As in the preceding proof, the matrix $A = [T(\mathbf{e_1}) \,|\, T(\mathbf{e_2})]$. To find $T(\mathbf{e_1})$ and $T(\mathbf{e_2})$, we must express $\mathbf{e_1}$ and $\mathbf{e_2}$ in terms of the two vectors whose images under T we do know. Observe that

$$
\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

And it can also be seen that

$$
\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

So due to the linearity of T:

$$
T(\mathbf{e_1}) = -T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix}
$$

$$T(\mathbf{e_2}) = 2T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + -T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Then the matrix A is $[T(\mathbf{e_1}) \mid T(\mathbf{e_2})]$:

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

6.1 Geometric Transformations in $\Re^2$

This section consists of various linear transformations that have a geometrical inter-
pretation. Such transformations form the building blocks for understanding linear
transformations.

Examples of Geometric Transformations

- Operators on $\Re^2$ and $\Re^3$ that map each vector onto its symmetric image about
  some line or plane are called **reflection operators**. There are three main
  reflections in $\Re^2$. Considering the transformation from the coordinates (x, y) to
  (w_1, w_2) the properties of each operator are as follows.

  1. **Reflection about the y-axis**: The equations for this transformation are

  $$w_1 = -x$$
  $$w_2 = y$$

  That is, $T(x) = -x$ and $T(y) = y$. The matrix A such that $T = T_A$ is

  $$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

  To demonstrate the reflection, we consider this example.

  Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

  2. **Reflection about the x-axis**: The equations for this transformation are

  $$w_1 = x$$
  $$w_2 = -y$$

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That is, $T(x) = x$ and $T(y) = -y$. The matrix A such that $T = T_A$ is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To demonstrate the reflection, we consider this example.

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3. **Reflection about the line** $y = x$: The equations for this transformation are

$$\begin{aligned} w_1 &= y \\ w_2 &= x \end{aligned}$$

That is, $T(x) = y$ and $T(y) = x$. The matrix A such that $T = T_A$ is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To demonstrate the reflection, we consider this example.

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- Operators on $\Re^2$ and $\Re^3$ that map each vector into its orthogonal projection on a line or plane through the origin are called **orthogonal projection operators**. There are two main projections in $\Re^2$. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of each operator are as follows.

1. **Orthogonal projection onto the** x**-axis**: The equations for this transformation are

$$\begin{aligned} w_1 &= x \\ w_2 &= 0 \end{aligned}$$

That is, $T(x) = x$ and $T(y) = 0$. The matrix A such that $T = T_A$ is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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To demonstrate the projection, we consider this example.

$$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \text{ Then } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2. **Orthogonal projection on the y-axis**: The equations for this transformation are

$$
\begin{aligned}
w_1 &= 0 \\
w_2 &= y
\end{aligned}
$$

That is, $T(x) = 0$ and $T(y) = y$. The matrix A such that $T = T_A$ is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To demonstrate the projection, we consider this example.

$$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \text{ Then } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- An operator that rotates each vector in $\Re^2$, through a fixed angle θ is called a **rotation operator** on $\Re^2$. There is only one rotation in $\Re^2$, due to the generality of the formula. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of the operator are as follows.

  Rotation through an angle θ: The equations for this transformation are

$$
\begin{aligned}
w_1 &= x \cos\theta - y \sin\theta \\
w_2 &= x \sin\theta + y \cos\theta
\end{aligned}
$$

That is, $T(x) = x \cos\theta - y \sin\theta$ and $T(y) = x \sin\theta + y \cos\theta$. The matrix A such that $T = T_A$ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To demonstrate the projection, we consider this example.

$$\text{Let } \theta = 30° \text{ and let } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ Then } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

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- If k is a nonnegative scalar, then the operator $T(\mathbf{x}) = k\mathbf{x}$ on $\Re^2$ and $\Re^3$ is called a **contraction with factor** k if $0 \le k \le 1$, and a **dilation with factor** k if $k \ge 1$. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of each operator are as follows.

  1. **Contraction with factor** k **on** $\Re^2$, $(0 \le k \le 1)$: The equations for this transformation are.

  $$w_1 = kx$$
  $$w_2 = ky$$

  That is, $T(x) = kx$ and $T(y) = ky$. The matrix A such that $T = T_A$ is

  $$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

  To demonstrate the contraction, we consider this example.

  Let $k = \dfrac{1}{2}$ and let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

  2. **Dilation with factor** k **on** $\Re^2$, $(k \ge 1)$: The equations for this transformation are

  $$w_1 = kx$$
  $$w_2 = ky$$

  The standard matrix for the transformation is clearly

  $$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

  To demonstrate the dilation, we consider this example.

  Let $k = 2$ and let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

6.2   Basic Properties of Linear Transformations

Theorem 6.2. *If $T : V \to W$ is a linear transformation, then:*

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

(c) $T(\mathbf{v} - \mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

Proof.   (a) Let $\mathbf{v}$ be any vector in V. Since $0\mathbf{v} = \mathbf{0}$, we have that

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$$

(b) Also,
$$T(-\mathbf{v}) = T((-1)\mathbf{v}) = (-1)T(\mathbf{v}) = -T(\mathbf{v})$$

(c) Finally, $\mathbf{v} - \mathbf{w} = \mathbf{v} + -1\mathbf{w}$; thus,

$$\begin{aligned}
T(\mathbf{v} - \mathbf{w}) &= T(\mathbf{v} + -1\mathbf{w}) \\
&= T(\mathbf{v})(-1)T(\mathbf{w}) \\
&= T(\mathbf{v}) - T(\mathbf{w})
\end{aligned}$$

$\square$

In words, part (a) of the Theorem 6.2 states that a linear transformation maps the zero vector in V onto the zero vector in W. This property is useful for identifying transformations that are not linear.

6.3   Product of Linear Transformations

Definition

- If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, the **composite of T_2 with T_1** denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

where $\mathbf{u}$ is a vector in U.

Remark: Observe that this definition requires the domain of T_2 (which is V) to contain the range of T_1; this is essential for the $T_2(T_1(\mathbf{u}))$ to be well-defined.

The next result shows that the composition of two linear transformations is itself a linear transformation.

Theorem 6.3. *If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.*

Proof. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in U and s and t are scalars, then it follows from the definition of a composite transformation and from the linearity of T_1 and T_2 that

$$
\begin{aligned}
T_2 \circ T_1(s\mathbf{u} + t\mathbf{v}) &= T_2(T_1(s\mathbf{u} + t\mathbf{v})) \\
&= T_2(sT_1(\mathbf{u}) + tT_1(\mathbf{v})) \\
&= sT_2(T_1(\mathbf{u})) + tT_2(T_1(\mathbf{v})) \\
&= sT_2 \circ T_1(\mathbf{u}) + tT_2 \circ T_1(\mathbf{v})
\end{aligned}
$$

$\square$

Examples

1. Let A be an $m \times n$ matrix, and B be an $n \times p$ matrix, then AB is an $m \times p$ matrix. Also $T_A : \Re^n \to \Re^m$, and $T_B : \Re^p \to \Re^n$ are both linear transformations. Then

$$
\begin{aligned}
T_A \circ T_B &= T_A(T_B(\mathbf{x})) \\
&= AB\mathbf{x} \\
&= (AB)\mathbf{x} \\
&= T_{AB}(\mathbf{x})
\end{aligned}
$$

where $\mathbf{x} \in \Re^p$. And therefore $T_A \circ T_B = T_{AB} : \Re^p \to \Re^m$.

2. If V has a basis $\beta = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $T : V \to V$ is a linear transformation given by

$$
\begin{aligned}
T(\mathbf{v_1}) &= 2\mathbf{v_1} + 3\mathbf{v_2} \\
T(\mathbf{v_2}) &= -7\mathbf{v_1} + 8\mathbf{v_2}
\end{aligned}
$$

To find $T \circ T(-\mathbf{v_1} + 3\mathbf{v_2})$ takes two steps as shown below.

$$
\begin{aligned}
T(-\mathbf{v_1} + 3\mathbf{v_2}) &= -T(\mathbf{v_1}) + 3T(\mathbf{v_2}) \\
&= -2\mathbf{v_1} - 3\mathbf{v_2} + 3(-7\mathbf{v_1} + 8\mathbf{v_2}) \\
&= -23\mathbf{v_1} + 21\mathbf{v_2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
T \circ T(-\mathbf{v_1} + 3\mathbf{v_2}) &= T(-23\mathbf{v_1} + 21\mathbf{v_2}) \\
&= -23T(\mathbf{v_1}) + 21T(\mathbf{v_2}) \\
&= -23(2\mathbf{v_1} + 3\mathbf{v_2}) + 21(-7\mathbf{v_1} + 8\mathbf{v_2}) \\
&= -193\mathbf{v_1} + 99\mathbf{v_2}
\end{aligned}
$$

6.4   Kernel and Image

Recall that if A is an $m \times n$ matrix, then the null space of A consists of all vectors $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x} = \mathbf{0}$. Also, by Theorem 5.7, the column space of A consists of all vectors $\mathbf{b}$ in $\Re^m$ for which there is at least one vector $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x} = \mathbf{b}$. From the viewpoint of linear transformations, the null space of A consists of all vectors in $\Re^n$ that map to $\mathbf{0}$ under T_A, and the column space of A consists of all vectors in $\Re^m$ that are images of at least one vector in $\Re^n$ under T_A. The following definition extends these ideas to general linear transformations.

Definitions

- If $T : V \to W$ is a linear transformation, then the set of vectors in V that map to $\mathbf{0}$ under T is called the **kernel** of T, denoted by $ker(T)$. Symbolically,

$$
ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}
$$

- If $T : V \to W$ is a linear transformation, then the set of all vectors in W that are images under T of at least one vector in V is called the **Image** (or range in some texts) of T; it is denoted by $Im(T)$. Symbolically,

$$
Im(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{u} \in V\}
$$

Examples

1. The kernel of the linear transformation $T_A(\mathbf{x}) = A\mathbf{x}$ is the null space of A, and the image of T_A is the column space of A.

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2. Let $I : V \to V$ be the identity operator. Since $I\mathbf{v} = \mathbf{v}$ for all vectors in V, every vector in V is the image of some vector (namely, itself); thus, $Im(I) = V$. Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, $ker(I) = \{\mathbf{0}\}$.

3. Let $T : \Re^3 \to \Re^3$ be the orthogonal projection onto the $x - y$ plane. The kernel of T is the set of points that T maps into $\mathbf{0} = (0, 0, 0)$; these are the points on the z-axis. Since T maps every point in $\Re^3$ onto the $x - y$ plane, the image of T must be some subset of this plane. But every point $(x_0, y_0, 0)$ in the $x - y$ plane is the image under T of some point; in fact, it is the image of all points on the line parallel to the z-axis, that passes through $(x_0, y_0, 0)$. Thus $Im(T)$ is the entire $x - y$ plane.

4. Let $T : \Re^2 \to \Re^2$ be the linear operator that rotates each vector in the $x - y$ plane through the angle θ. Since *every* vector in the $x - y$ plane can be obtained by rotating some vector through the angle θ, $Im(T) = \Re^2$. Moreover, the only vector that rotates onto $\mathbf{0}$ is $\mathbf{0}$, so $ker(T) = \{\mathbf{0}\}$.

5. Find the kernel of the linear transformation $T : \Re^3 \to \Re^3$ given by

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

It is thus necessary to find the vectors $[x, y, z]^T$ such that

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x = y = 0$ and z is an arbitrary real number. Therefore the set of vectors of the form

$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

with z arbitrary is equal to the the kernel of T. (The standard basis vector $\mathbf{e_3}$ forms a basis for $ker(T)$.)

In all of the preceding examples, $ker(T)$ and $Im(T)$ turned out to be *subspaces*. This is no accident, as we see in the following theorem.

Theorem 6.4. *If $T : V \to W$ is a linear transformation, then:*

(a) The kernel of T is a subspace of V.

(b) The range of T is a subspace of W.

Proof.   (a) For $ker(T)$ to be a subspace, it must contain at least one vector and be closed under addition and scalar multiplication. By part (a) of Theorem 6.2, the vector $\mathbf{0}$ is in $ker(T)$, so $ker(T)$ is non-empty. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be vectors in $ker(T)$, and let k be any scalar. Then

$$T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so that $\mathbf{v_1} + \mathbf{v_2} \in ker(T)$. Also,

$$T(k\mathbf{v_1}) = kT(\mathbf{v_1}) = k\mathbf{0} = \mathbf{0}$$

so that $k\mathbf{v_1} \in ker(T)$.

(b) Since $T(\mathbf{0}) = \mathbf{0}$, there is at least one vector in $Im(T)$. Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be vectors in the image of T, and let k be any scalar. Since $\mathbf{w_1}$ and $\mathbf{w_2}$ are in the image of T, there are vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ in V such that $T(\mathbf{a_1}) = \mathbf{w_1}$ and $T(\mathbf{a_2}) = \mathbf{w_2}$. Let $\mathbf{a} = \mathbf{a_1} + \mathbf{a_2}$ and $\mathbf{b} = k\mathbf{a_1}$. Then

$$T(\mathbf{a}) = T(\mathbf{a_1} + \mathbf{a_2}) = T(\mathbf{a_1}) + T(\mathbf{a_2}) = \mathbf{w_1} + \mathbf{w_2}$$

and

$$T(\mathbf{b}) = T(k\mathbf{a_1}) = kT(\mathbf{a_1}) = k\mathbf{w_1}$$

Thus $\mathbf{w_1} + \mathbf{w_2}$ and $k\mathbf{w_1}$ are in the image of T, and $Im(T)$ is closed under addition and scalar multiplication.

$\square$

Theorem 6.5. *If $T : U \to V$ is a linear transformation and $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ forms a basis for U, then $Im(T) = span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n}))$*

Example

Let A be an $m \times n$ matrix and let $T = T_A : \Re^n \to \Re^m$. Let $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ be the standard basis for $\Re^n$. Then by the previous theorem

$$
\begin{aligned}
Im(T_A) &= span(T_A(\mathbf{e_1}), T_A(\mathbf{e_2}), \ldots, T_A(\mathbf{e_n})) \\
&= span(A\mathbf{e_1}, A\mathbf{e_2}, \ldots, A\mathbf{e_n}) \\
&= span(\mathbf{c_1}(A), \mathbf{c_2}(A), \ldots, \mathbf{c_n}(A)) \\
&= C(A)
\end{aligned}
$$

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6.5 Rank and Nullity

In Section 5 of this course, the rank of a matrix was defined to be the dimension of its column (or row) space and the nullity to be the dimension of its null space. The following definition extends these definitions to general linear transformations.

Definitions If $T : V \to W$ is a linear transformation,

- then the dimension of $Im(T)$ is called the **rank of** T, denoted by rank(T),

- and the dimension of $ker(T)$ is called the **nullity of** T, denoted by nullity(T).

Examples

- Let U be a vector space of dimension n, with basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$, and let $T : U \to V$ be a linear transformation defined by

$$T(\mathbf{u_1}) = \mathbf{u_2}, T(\mathbf{u_2}) = \mathbf{u_3}, \cdots, T(\mathbf{u_{n-1}}) = \mathbf{u_n} \text{ and } T(\mathbf{u_n}) = \mathbf{0}$$

Find bases for $ker(T)$ and $Im(T)$ and determine rank(T) and nullity(T).

Using the result from Theorem 6.5 it can be stated

$$
\begin{aligned}
Im(T) &= span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})) \\
&= span(\mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_n})
\end{aligned}
$$

So $\mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_n}$ form a basis for $Im(T)$ and hence the rank$(T) = n - 1$.

Since $T(\mathbf{u_n}) = \mathbf{0}$, $\mathbf{u_n} \in ker(T)$. So $span(\mathbf{u_n}) \subset ker(T)$. Conversely, suppose that $\mathbf{u} \in ker(T)$. Then $T(\mathbf{u}) = \mathbf{0}$. As $\mathbf{u}$ is an element of U it can be expressed as a linear combination of the basis vectors of U, say as

$$\mathbf{u} = x_1 \mathbf{u_1} + x_2 \mathbf{u_2} + \cdots + x_n \mathbf{u_n}$$

Therefore
$$T(\mathbf{u}) = \mathbf{0} = x_1 T(\mathbf{u_1}) + x_2 T(\mathbf{u_2}) + \cdots + x_n T(\mathbf{u_n})$$

$$= x_1 \mathbf{u_2} + x_2 \mathbf{u_3} + \cdots + x_{n-1} \mathbf{u_{n-1}} + \mathbf{0}$$

Due to the linear independence of $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$, $x_1, x_2, \ldots, x_{n-1} = 0$. Therefore

$$\mathbf{u} = x_n \mathbf{u_n}$$

So $ker(T) \subseteq span(\mathbf{u_n})$ and $ker(T) = span(\mathbf{u_n})$. Thus nullity$(T) = 1$.

If A is an $m \times n$ matrix and $T_A : \Re^n \to \Re^m$ is the transformation $T_A(\mathbf{x}) = A\mathbf{x}$, then from a previous example, $ker(T_A) = N(A)$ and $Im(T_A) = C(A)$. Thus, the following relationship exists between the rank and nullity of a matrix A and the rank and nullity of the corresponding linear transformation T_A.

Theorem 6.6. *If A is an $m \times n$ matrix and $T_A : \Re^n \to \Re^m$ is multiplication by A, then:*

(a) *nullity(T_A) = nullity(A)*

(b) *rank(T_A) = rank(A)*

Recall from (Rank Plus Nullity) Theorem 5.6 that if A is a matrix with n columns, then rank(A) + nullity(A) = n. The following theorem extends this result to general linear transformations.

Theorem 6.7. *If $T : V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then*

$$rank(T) + nullity(T) = dim(V) = n$$

Proof. The proof is divided into two cases.

Case 1 Let V be the zero vector space (dim(V)=0). By Theorem 6.2, $T(\mathbf{0}) = \mathbf{0}$. $Im(T) = \{\mathbf{0}\}$ and $ker(T) = \{\mathbf{0}\}$ giving

$$rank(T) + nullity(T) = 0 + 0 = 0 = dim(V).$$

Case 2 Let V be an n-dimensional vector space with the basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$. The proof can be divided up into three parts.

(a) $ker(T) = \{\mathbf{0}\}$
Let $\mathbf{u} \in ker(T)$. As $\mathbf{u} \in V$, it can be expressed as a linear combination of the basis vectors, say as

$$\mathbf{u} = x_1\mathbf{u_1} + x_2\mathbf{u_2} + \cdots + x_n\mathbf{u_n} \tag{1}$$

As $\mathbf{u} \in ker(T)$,

$$T(\mathbf{u}) = \mathbf{0} = x_1 T(\mathbf{u_1}) + x_2 T(\mathbf{u_2}) + \cdots + x_n T(\mathbf{u_n}) \tag{2}$$

Since $ker(T) = \{\mathbf{0}\}$, $\mathbf{u} = \mathbf{0}$. From equation (1), the linear independence of $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ gives that $x_1, x_2, \ldots, x_n = 0$. It then follows from equation (2) that $T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})$ are linearly independent. It is known from Theorem 6.4 that $Im(T) = span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n}))$. As $T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})$ are linearly independent they form a basis for $Im(T)$. It can therefore be stated that

$$\text{rank}(T) + \text{nullity}(T) = n + 0 = n = \dim(U)$$

(b) $ker(T) = V$

Theorem 6.4 states that $Im(T) = span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n}))$. However $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n} \in \ker(T)$. Therefore $T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n}) = \mathbf{0}$. So $Im(T) = span(\mathbf{0}) = \{\mathbf{0}\}$. Therefore

$$\text{rank}(T) + \text{nullity}(T) = 0 + n = n = \dim(V).$$

(c) $1 \leq \text{nullity}(T) < n$

Assume that the $\text{nullity}(T) = r$, and let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ be a basis for the kernel. Since $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ form a linearly independent set, Theorem 3.6(b) states that there are $n - r$ vectors, $\mathbf{u_{r+1}}, \mathbf{u_{r+2}}, \ldots, \mathbf{u_n}$, such that $\{\mathbf{u_1}, \ldots, \mathbf{u_r}, \mathbf{u_{r+1}}, \ldots, \mathbf{u_n}\}$ is a basis for V. To complete the proof it shall be shown that the $n - r$ vectors in the set $S = \{T(\mathbf{u_{r+1}}), \ldots, T(\mathbf{u_n})\}$ form a basis for the image of T.

First it shall be shown that S spans the image of T. If $\mathbf{b}$ is any vector in the range of T, the $\mathbf{b} = T(\mathbf{u})$ for some vector $\mathbf{u}$ in V. Since $\{\mathbf{u_1}, \ldots, \mathbf{u_r}, \mathbf{u_{r+1}}, \ldots, \mathbf{u_n}\}$ is a basis for V, the vector $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1 \mathbf{u_1} + \cdots + c_r \mathbf{u_r} + c_{r+1} \mathbf{u_{r+1}} + \cdots + c_n \mathbf{u_n}.$$

Since $\mathbf{u_1}, \ldots, \mathbf{u_r}$ lie in the kernel of T, $T(\mathbf{u_1}), \ldots, T(\mathbf{u_r}) = \mathbf{0}$, so that

$$\mathbf{b} = T(\mathbf{u}) = c_{r+1} T(\mathbf{u_{r+1}}) + \cdots + c_n T(\mathbf{u_n})$$

Thus, S spans the image of T.

We now show that S is a linearly independent set and consequently forms a basis for the image of T. Suppose that

$$k_{r+1} T(\mathbf{u_{r+1}}) + \cdots + k_n T(\mathbf{u_n}) = \mathbf{0} \tag{3}$$

It must be shown that $k_{r+1} = \cdots = k_n = 0$. Since T is linear, equation (3) can be rewritten as

$$T(k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n}) = \mathbf{0}$$

which says that $k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n}$ is in the kernel of T. This vector can therefore be written as a linear combination of the basis vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$, say

$$k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n} = k_1\mathbf{u_1} + \cdots + k_r\mathbf{u_r}$$

Thus,

$$k_1\mathbf{u_1} + \cdots + k_r\mathbf{u_r} - k_{r+1}\mathbf{u_{r+1}} - \cdots - k_n\mathbf{u_n} = \mathbf{0}$$

Since $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ is linearly independent, all of the k_i's are zero; in particular $k_{r+1} = \cdots = k_n = 0$. Thus S is a basis for the image of T. Thus

$$\text{rank}(T) + \text{nullity}(T) = (n - r) + r = n = \dim(V).$$

$\square$

Example Let $T : \Re^2 \to \Re^2$ be the linear operator that rotates each vector in the $x - y$ plane through an angle of θ. It was showed previously that $ker(T) = \{\mathbf{0}\}$ and $Im(T) = \Re^2$. Thus,

$$\text{rank}(T) + \text{nullity}(T) = 2 + 0 = 2 = \dim(\Re^2).$$

6.6 Matrix of a Linear Transformation

In this section it shall be shown that if V and W are finite-dimensional vector spaces, then with a little ingenuity any linear transformation $T : V \to W$ can be regarded as a matrix transformation. The basic idea is to work with coordinate vectors rather than with the vectors themselves.

Definition

- Suppose that V is an n-dimensional vector space and W an m-dimensional vector space, and let $T : V \to W$ be a linear transformation. Let β and γ be bases for V and W respectively. Then for each $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_\beta$ will be a vector in $\Re^n$, and the coordinate vector $[T(\mathbf{x})]_\gamma$ will be a vector in $\Re^m$. If there exists an $m \times n$ matrix A, such that

$$A[\mathbf{x}]_\beta = [T(\mathbf{x})]_\gamma \tag{4}$$

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then A is called **the matrix of the transformation relative to bases** β **and** γ and it is written

$$A = [T]_\beta^\gamma$$

Theorem 6.8. *Let* $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ *and* $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ *be bases for the vector spaces* V *and* W *respectively, and let* $\mathbf{x} \in V$. *If* $T : V \to W$ *is a linear transformation then*

(a) *the matrix of transformation relative to bases* β *and* γ *always exists. That is to say, there always exists a matrix* $A = [T]_\beta^\gamma$ *such that*

$$A[\mathbf{x}]_\beta = [T(\mathbf{x})]_\gamma$$

(b) *The matrix of transformation relative to bases* β *and* γ *has the form*

$$[T]_\beta^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma \mid \cdots \mid [T(\mathbf{u_n})]_\gamma]$$

Proof. Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ be a basis for the n-dimensional space V and let $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ be a basis for the m-dimensional space W. Then the matrix $[T]_\beta^\gamma = A$ must have the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

such that equation (4) holds for all vectors $\mathbf{x}$ in V. In particular, this equation must hold for the basis vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$; that is,

$$A[\mathbf{u_1}]_\beta = [T(\mathbf{u_1})]_\gamma, A[\mathbf{u_2}]_\beta = [T(\mathbf{u_2})]_\gamma, \ldots, A[\mathbf{u_n}]_\beta = [T(\mathbf{u_n})]_\gamma \qquad (5)$$

But

$$[\mathbf{u_1}]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, [\mathbf{u_2}]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, [\mathbf{u_n}]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

so

$$A[\mathbf{u_1}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$A[\mathbf{u_2}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vdots$$

$$A[\mathbf{u_n}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these results into equation (4) yields

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(\mathbf{u_1})]_\gamma, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = [T(\mathbf{u_2})]_\gamma, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(\mathbf{u_n})]_\gamma$$

which shows that the columns of A are (in order) the coordinate vectors of $T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})$ with respect to the basis γ. Thus the matrix of the transformation T with respect to the bases β and γ always exists and is given by

$$[T]_\beta^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma \mid \cdots \mid [T(\mathbf{u_n})]_\gamma]$$

$\square$

Examples

1. Let $T_B : \Re^n \to \Re^m$ be a linear transformation defined by $T(X) = BX$ where B is an $m \times n$ matrix. Let $\beta = \{\mathbf{E_1}, \mathbf{E_2}, \ldots, \mathbf{E_n}\}$ be the standard basis for $\Re^n$ and

let $\gamma = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}\}$ be the standard basis for $\Re^m$. Then it is known from the previous theorem that $[T]_\beta^\gamma$ is the following matrix

$$[T_B]_\beta^\gamma = [[T_B(\mathbf{E_1})]_\gamma \mid [T_B(\mathbf{E_2})]_\gamma \mid \cdots \mid [T_B(\mathbf{E_n})]_\gamma]$$

For $1 \leq j \leq n$ it follows from the definition of the transformation that

$$T_B(\mathbf{E_j}) = B\mathbf{E_j} = col_j(B) = b_{1j}\mathbf{e_1} + b_{2j}\mathbf{e_2} + \cdots + b_{mj}\mathbf{e_m}$$

therefore

$$[T_B(\mathbf{E_j})]_\gamma = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = col_j(B)$$

and so $[T_B]_\beta^\gamma = B$.

2. Let U have the basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and let V have the basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$. Let T be the linear transformation defined by

$$T(\mathbf{u_1}) = 2\mathbf{v_1} + \mathbf{v_2}, \ T(\mathbf{u_2}) = \mathbf{v_1} - \mathbf{v_2}, \ T(\mathbf{u_3}) = 2\mathbf{v_2}$$

Then

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

3. Let $V = M_{2\times 2}(\Re)$ and let $T : V \to V$ be the linear transformation given by $T(X) = BX - XB$ where $X \in V$ and

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for V; that is,

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To find $[T]_\beta^\beta$ it is necessary to do the following calculations:

$$T(E_{11}) = BE_{11} - E_{11}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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$$= \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = 0E_{11} + -bE_{12} + cE_{21} + 0E_{22}$$

$$T(E_{12}) = BE_{12} - E_{12}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix} = -cE_{11} + (a-d)E_{12} + 0E_{21} + cE_{22}$$

$$T(E_{21}) = BE_{21} - E_{21}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} b & 0 \\ d-a & -b \end{bmatrix} = bE_{11} + 0E_{12} + (d-a)E_{21} + -bE_{22}$$

$$T(E_{22}) = BE_{22} - E_{22}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} = 0E_{11} + bE_{12} + -cE_{21} + 0E_{22}$$

So the matrix of the linear operator is

$$[T]_{\beta}^{\beta} = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{bmatrix}$$

The following theorem follows directly from the definition of the matrix of a linear transformation.

Theorem 6.9. *Let $T : V \to W$ be a linear transformation, and let β and γ be bases for V and W respectively. Then if $\mathbf{v} \in V$*

$$[T(\mathbf{v})]_{\gamma} = [T]_{\beta}^{\gamma}[\mathbf{v}]_{\beta}$$

Example

1. Let U have the basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and let V have the basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$. Let T be the linear transformation defined by

$$T(\mathbf{u_1}) = 2\mathbf{v_1} + \mathbf{v_2}, \ T(\mathbf{u_2}) = \mathbf{v_1} - \mathbf{v_2}, \ T(\mathbf{u_3}) = 2\mathbf{v_2}$$

Given that $\mathbf{u} = 3\mathbf{u_1} + -2\mathbf{u_2} + 7\mathbf{u_3}$

$$[\mathbf{u}]_\beta = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}, \text{ and } [T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Hence

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \end{bmatrix}$$

Hence $T(\mathbf{u}) = 4\mathbf{v_1} + 19\mathbf{v_2}$.

The following theorem gives a recipe for finding bases for $ker(T)$ and $Im(T)$. The proof is omitted, but the result is demonstrated in the example following.

Theorem 6.10. *Let A be the $m \times n$ matrix such that $A = [T]_\beta^\gamma$, where $T : V \to W$ is a linear transformation, and $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $\gamma = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}\}$ are bases for V and W respectively. Let $s = nullity(A)$ and $r = rank(A)$. Then suppose that*

$$\mathbf{x_j} = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix}$$

for $1 \le j \le s$, form a basis for $N(A)$, while $col_{c_1}(A), col_{c_2}(A), \ldots, col_{c_r}(A)$ form a basis for $C(A)$. Then

1. *(a) the vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_s}$ defined by*

$$\mathbf{u_j} = x_{1j}\mathbf{v_1} + x_{2j}\mathbf{v_2} + \cdots + x_{nj}\mathbf{v_n}$$

   will be a basis for the kernel of T.

   (b) the vectors $T(\mathbf{v_{c_1}}), T(\mathbf{v_{c_2}}), \ldots, T(\mathbf{v_{c_r}})$ form a basis for the image of T.

2. *If $N(A) = \{\mathbf{0}\}$, then $ker(T) = \{\mathbf{0}\}$*
   If $C(A) = \{\mathbf{0}\}$, then $Im(T) = \{\mathbf{0}\}$

3. *$rank(T) = rank(A)$ and $nullity(T) = nullity(A)$.*

Example

1. Let $T : V \to W$ be a linear transformation, and let $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $\gamma = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be bases for V and W respectively. T is the linear transformation given by

$$
\begin{aligned}
T(\mathbf{v_1}) &= \mathbf{w_1} + \mathbf{w_2} + -\mathbf{w_3} \\
T(\mathbf{v_2}) &= 2\mathbf{w_1} + -3\mathbf{w_2} \\
T(\mathbf{v_3}) &= 3\mathbf{w_1} + -2\mathbf{w_2} + -\mathbf{w_3}
\end{aligned}
$$

Let $A = [T]_\beta^\gamma$. Therefore

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 0 & -1 \end{bmatrix}
$$

Let $B = rref(A)$, then

$$
B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

Any solution vector is a scalar multiple of the vector

$$
\mathbf{x_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
$$

and this vector is thus a basis for $N(A)$. It follows from Theorem 6.10 that $-\mathbf{v_1} - \mathbf{v_2} + \mathbf{v_3}$ is a basis for the kernel of T.

It can also be seen that $col_1(A), col_2(A)$ form a basis for $C(A)$. So $T(\mathbf{v_1})$ and $T(\mathbf{v_2})$ form a basis for $Im(T)$.

6.7   Similar Matrices

The matrix of a linear operator $T : V \to V$ depends on the basis selected for V. One of the fundamental problems of linear algebra is to choose a basis for V that makes the matrix for T as simple as possible - diagonal or triangular, for example. This section is devoted to the study of this problem.

To demonstrate that certain bases produce a much simpler matrix of transformation than others, consider the following example.

Example

1. Standard bases do not necessarily produce the simplest matrices for linear operators. For example, consider the linear operator $T : \Re^2 \to \Re^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

and let $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ be the standard basis for $\Re^2$. By Theorem 6.8, the matrix for T with respect to this basis is

$$[T]_\beta^\beta = [[T(\mathbf{e_1})]_\beta \mid [T(\mathbf{e_2})]_\beta] = [T(\mathbf{e_1}) \mid T(\mathbf{e_2})]$$

(we shall refer to the matrix for a transformation T with respect to the standard basis as the **standard matrix for** T.) From the definition of the linear transformation T,

$$T(\mathbf{e_1}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{so} \quad [T]_\beta^\beta = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

In comparison, consider the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

By Theorem 6.8, the matrix for T with respect to the basis γ is

$$[T]_\gamma^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma$$

From the definition of the linear transformation T,

$$T(\mathbf{u_1}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{u_1}, \quad T(\mathbf{u_2}) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{u_2}$$

Hence

$$[T(\mathbf{u_1})]_\gamma = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u_2})]_\gamma = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \text{and} \quad [T]_\beta^\beta = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

This matrix is 'simpler' in the sense that diagonal matrices enjoy special properties that arbitrary matrices do not.

Much research has been devoted to determining the 'simplest possible form' that can be obtained for the matrix of a linear operator $T : V \to V$, by choosing the basis for V appropriately. This problem can be attacked by first finding a matrix for T relative to *any* basis, say a standard basis, where applicable, then changing the basis in a manner that simplifies the matrix. Before pursuing this idea further, it is necessary to grasp the theorem below. It gives a useful alternative viewpoint about change of basis matrices; it shows that the change of basis matrix from a basis β to γ can be regarded as the matrix of transformation of the identity operator.

Theorem 6.11. *If β and γ are bases for a finite-dimensional vector space V, and if $I : V \to V$ is the identity operator, then $[I]_{\beta}^{\gamma}$ is the change of basis matrix from β to γ.*

Proof. Suppose that $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are bases for V. Using the fact that $I(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$, it follows that

$$
\begin{aligned}
[I]_{\beta}^{\gamma} &= [[I(\mathbf{u_1})]_{\gamma} \mid [I(\mathbf{u_2})]_{\gamma} \mid \cdots \mid [I(\mathbf{u_n})]_{\gamma}] \\
&= [[\mathbf{u_1}]_{\gamma} \mid [\mathbf{u_2}]_{\gamma} \mid \cdots \mid [\mathbf{u_n}]_{\gamma}]
\end{aligned}
$$

which by Theorem 3.8 is the change of basis matrix from β to γ. $\qquad\square$

Problem: If β and γ are two bases for a finite-dimensional vector space V, and if $T : V \to V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_{\beta}^{\beta}$ and $[T]_{\gamma}^{\gamma}$?

To answer this question we consider the composition of three linear operators. Let $\mathbf{v}$ be a vector in V. Let $\mathbf{v}$ be mapped onto itself by the identity operator, then let $\mathbf{v}$ be mapped onto $T(\mathbf{v})$ by T, then let $T(\mathbf{v})$ be mapped onto itself by the identity operator. All four vector spaces involved in the composition are the same (namely V); however, the bases for the spaces can vary. Since the starting vector is $\mathbf{v}$ and the final vector is $T(\mathbf{v}$, the composition is the same as T; that is,

$$T = I \circ T \circ I$$

If the first and last vector spaces are assigned the basis γ and the middle two spaces are assigned the basis β, then it follows from the previous statement $T = I \circ T \circ I$, that

$$[T]_{\gamma}^{\gamma} = [I \circ T \circ I]_{\gamma}^{\gamma} = [I]_{\beta}^{\gamma}[T]_{\beta}^{\beta}[I]_{\gamma}^{\beta}$$

But it follows from Theorem 6.11 that $[I]_\gamma^\beta$ is the change of basis matrix from γ to β and consequently $[I]_\beta^\gamma$ is the change of basis matrix from β to γ. Thus, let $P = [I]_\gamma^\beta$, then $P^{-1} = [I]_\beta^\gamma$ and hence

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P$$

Theorem 6.12. *Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let β and γ be bases for V. Then*

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P \tag{6}$$

where P is the change of basis matrix from γ to β.

Remark: When applying Theorem 6.12, it is easy to forget whether P is the change of basis matrix from β to γ or the change of basis matrix from γ to β. Just remember that in order for $[T]_\beta^\beta$ to operate successfully on a vector $\mathbf{v}$, $\mathbf{v}$ must be expressed in terms of the basis β. Therefore, due to P's positioning in the formula, it must be the change of basis matrix from γ to β.

Example

1. Let $T : \Re^2 \to \Re^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

Find the matrix of T with respect to the standard basis $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ for $\Re^2$, then use Theorem 6.12 to find the matrix of T with respect to the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

It was shown earlier that

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

To find $[T]_\gamma^\gamma$ from (6), we require the change of basis matrix P, where

$$P = [I]_\gamma^\beta = [[\mathbf{u_1}]_\beta \mid [\mathbf{u_2}]_\gamma]$$

By inspection

$$\mathbf{u_1} = \mathbf{e_1} + \mathbf{e_2}$$
$$\mathbf{u_2} = \mathbf{e_1} + 2\mathbf{e_2}$$

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so that
$$[\mathbf{u_1}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [\mathbf{u_2}]_\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The inverse of P is
$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

so that by Theorem 6.12 the matrix of T relative to the basis γ is

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

which agrees with the result from the previous example.

The relationship in (6) is of such importance that there is some terminology associated with it.

Definition

- If A and B are square matrices, it is said that B **is similar to** A if there is an invertible matrix P such that $B = P^{-1}AP$.

It follows from the definition that *two matrices representing the same linear operator $T : V \to V$ with respect to different bases are similar.* The following example demonstrates just this phenomenon. It helps to remember that a change of basis in $\Re^2$ or $\Re^3$ really just amounts to a change in the coordinate system. When reading the following example, note how the choice of the coordinate system can greatly simplify the form of the matrix of the transformation.

Example

1. Let l be the line in the $x - y$ plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \le \theta \le \pi$. Let $T : \Re^2 \to \Re^2$ be the linear operator that maps each vector into its reflection about the line l.

   (a) Find the standard matrix for T. (Remember that the standard matrix is the matrix of the linear operator relative to the standard basis.)

   (b) Find the reflection of the vector $\mathbf{x} = (1, 2)$ about the line l through the origin that makes an angle of $\theta = \frac{\pi}{6}$ with the positive x-axis.

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Solution:

(a) This problem could be solved by trying to construct the standard matrix from the formula
$$[T]^\beta_\beta = [T(\mathbf{e_1}) \mid T(\mathbf{e_2})]$$
where $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ is the standard basis for $\Re^2$. However, it is easier to use a different strategy: Instead of finding $[T]^\beta_\beta$ directly, it is simpler to first find the matrix $[T]^\gamma_\gamma$ where

$$\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$$

is the basis of a unit vector $\mathbf{u_1}$ along l and a unit vector $\mathbf{u_2}$ perpendicular to l. Once $[T]^\gamma_\gamma$ has been found, it is simply a matter of using Theorem 6.12 to perform a change of basis to find $[T]^\beta_\beta$. The computations are as follows:
$$T(\mathbf{u_1}) = \mathbf{u_1} \text{ and } T(\mathbf{u_2}) = -\mathbf{u_2}$$

so
$$[T(\mathbf{u_1})]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [T(\mathbf{u_2})]_\gamma = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Thus,
$$[T]^\gamma_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is clear from the geometry of the problem that

$$[\mathbf{u_1}]_\beta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } [\mathbf{u_2}]_\beta = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore the change of basis matrix from β to γ is

$$P = \begin{bmatrix} [\mathbf{u_1}]_\gamma & \mid & [\mathbf{u_2}]_\gamma \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

It follows from equation (6) that

$$[T]^\beta_\beta = P[T]^\gamma_\gamma P^{-1}$$

Thus, the standard matrix for T is

$$[T]_\beta^\beta = P[T]_\gamma^\gamma P^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

(b) It follows from part (a) that the formula for T in matrix notation is

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Substituting $\theta = \frac{\pi}{6}$ in this formula yields

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \sqrt{3} \\ \frac{\sqrt{3}}{2} - 1 \end{bmatrix}$$

Thus, $T([1, 2]) = [\frac{1}{2} + \sqrt{3}, \frac{\sqrt{3}}{2} - 1]$.

It has already been stated that the selection of the correct basis can result in a simpler form of the matrix of linear transformation $T : V \to V$. In the previous example, the selection of the basis γ resulted in a diagonal matrix of transformation. Given the computational advantages of diagonal matrices (refer to section 1.6.1), it is important to find ways (if possible) to choose a basis for V such that the transformation matrix is diagonal. This problem can be stated in another way. Given an $n \times n$ matrix of transformation $[T]_\beta^\beta$, what change of basis matrix P is required such that $P^{-1}[T]_\beta^\beta P$ is a diagonal matrix. This very problem is attacked in the next section.

7 EIGENVALUES AND EIGENVECTORS

This section will be concerned with the following two problems.

1. **The Diagonalization Problem (matrix form).** Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix? Or to state it in another way: If A is an $n \times n$ matrix, does there exist a diagonal matrix D such that A is similar to D?

2. **The Eigenvalue Problem:** If A is an $n \times n$ matrix, are there nonzero vectors $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

These two problems are very closely related. In fact, a solution to the eigenvalue problem leads to a solution of the diagonalization problem. We first consider the eigenvalue problem.

7.1 The Eigenvalue Problem

Definition

- If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\Re^n$ is called an **eigenvector** of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

  for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an eigenvector of A **corresponding** to λ.

Examples - *VERIFYING EIGENVALUES AND EIGENVECTORS*

1. For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

   verify that $\mathbf{x_1} = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and $\mathbf{x_2} = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

   Multiplying A by $\mathbf{x_1}$ produces

$$A\mathbf{x_1} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{x_1} = \lambda_1\mathbf{x_1}$$

Thus $\mathbf{x_1} = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$.

Similarly, multiplying A by $\mathbf{x_2}$ produces

$$A\mathbf{x_2} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1\mathbf{x_2} = \lambda_2 \mathbf{x_2}$$

Thus $\mathbf{x_2} = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

2. For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that $\mathbf{x_1} = (-3, -1, 1)$ and $\mathbf{x_2} = (1, 0, 0)$ are eigenvectors of A and find their corresponding eigenvalues.

Multiplying A by $\mathbf{x_1}$ produces

$$A\mathbf{x_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{x_1} = \lambda_1 \mathbf{x_1}$$

Thus $\mathbf{x_1} = (-3, -1, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.

Similarly, multiplying A by $\mathbf{x_2}$ produces

$$A\mathbf{x_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x_2} = \lambda_2 \mathbf{x_2}$$

Thus $\mathbf{x_2} = (1, 0, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

7.1.1   Determining the Eigenvalues

Theorem 7.1. *Let A be $n \times n$, $\lambda \in \Re$. Then λ is an eigenvalue of A if and only if it is a solution to the equation*

$$det(\lambda I_n - A) = 0$$

Proof. Assume $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \in \Re^n$, then

$$A\mathbf{x} = \lambda\mathbf{x} = I_n \lambda\mathbf{x} = \lambda I_n \mathbf{x}$$

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$$\therefore \lambda I_n \mathbf{x} - A\mathbf{x} = \mathbf{0}$$

$$(\lambda I_n - A)\mathbf{x} = \mathbf{0}$$

Since $\mathbf{x} \neq \mathbf{0}$, this last equation gives that the matrix $(\lambda I_n - A)$ cannot have an inverse (otherwise $\mathbf{x} = \mathbf{0}$ would be the one and only solution). Thus

$$det(\lambda I_n - A) = 0$$

The values of λ that satisfy this condition are the eigenvalues of the matrix A.   □

Remark: Note that $det(A - \lambda I_n) = 0$ also completely characterises the set of eigenvalues.

Definitions

- The equation $det(\lambda I_n - A) = 0$ is called the **characteristic equation** of A.

- The polynomial $det(xI_n - A)$ is called the **characteristic polynomial** of A, and is denoted by $ch_A(x)$.

- If λ is an eigenvalue of A and

$$ch_A(x) = (x - \lambda)^t \cdot g(x), \ g(\lambda) \neq 0$$

  then t is called the **algebraic multiplicity** of λ and is denoted by $a_A(\lambda)$.

Examples

1. Let A be the general $n \times n$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the characteristic polynomial of A is given by

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x - a_{nn} \end{vmatrix}$$

2. Let A be the 2×2 matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x - 2 & -1 \\ -1 & x - 2 \end{vmatrix} = (x - 2)^2 - 1 = (x - 1)(x - 3)$$

So the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3$. Note also that $a_A(1) = 1$ and $a_A(3) = 1$.

3. Let A be the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x - 1 & 0 & 0 & 0 \\ 0 & x - 2 & 0 & 0 \\ 0 & 0 & x - 3 & 0 \\ 0 & 0 & 0 & x - 4 \end{vmatrix}$$

$$= (x - 1)(x - 2)(x - 3)(x - 4)$$

So the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$. Note also that $a_A(1) = 1$, $a_A(2) = 1$, $a_A(3) = 1$ and $a_A(4) = 1$.

Remark: The previous problem reveals another property of diagonal matrices. The eigenvalues of a diagonal matrix are always equal to the entries on the diagonal. (Prove it!)

7.1.2 Determining the Eigenspace

The following theorem characterises the eigenvectors of a matrix A corresponding to an eigenvalue λ.

Theorem 7.2. *Let A be an $n \times n$ matrix and λ be an eigenvalue of A. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$.*

If A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A corresponding to λ. To see this, let c be a nonzero scalar:

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

Thus $c\mathbf{x}$ is also an eigenvector of A. It is also true that if $\mathbf{x_1}$ and $\mathbf{x_2}$ are eigenvectors to the *same* eigenvalue λ, then their sum is also an eigenvector corresponding to λ:

$$A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \lambda\mathbf{x_1} + \lambda\mathbf{x_2} = \lambda(\mathbf{x_1} + \mathbf{x_2})$$

Thus the set of all eigenvectors of a given eigenvalue λ, along with the zero vector, form a subspace of $\Re^n$.

Definitions

- If λ is an eigenvalue of A, the subspace $N(\lambda I_n - A)$ is called the **eigenspace** of A corresponding to λ. It is written $E_A(\lambda) = N(\lambda I_n - A)$.

- The dimension of $E_A(\lambda)$ is called the **geometric multiplicity** of λ and is denoted by $g_A(\lambda) = \text{nullity}(\lambda I_n - A)$.

Examples - *FINDING EIGENVALUES AND EIGENVECTORS*

1. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

   Also find the geometric multiplicity of each eigenvalue.

   The characteristic equation of A is given by

$$
\begin{aligned}
|\lambda I_2 - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\
&= (\lambda - 2)(\lambda + 5) + 12 \\
&= \lambda^2 + 3\lambda - 10 + 12 \\
&= \lambda^2 + 3\lambda + 2 \\
&= (\lambda + 2)(\lambda + 1) = 0,
\end{aligned}
$$

   which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A. Both eigenvalues have an algebraic multiplicity of one. To find the corresponding eigenvectors, Gaussian elimination is used to solve the homogenous linear system $(\lambda I_2 - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I_2 - A = \begin{bmatrix} -1-2 & 12 \\ -1 & -1+5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$$

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which row-reduces to

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

Therefore $x_1 - 4x_2 = 0$. Letting $x_2 = t$ (an arbitrary scalar), every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

So $E_A(-1) = \text{span}([4 \quad 1]^T)$ and $g_A(-1) = 1$.

For $\lambda_2 = -2$, the coefficient matrix is

$$(-2)I_2 - A = \begin{bmatrix} -2-2 & 12 \\ -1 & -2+5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

Therefore $x_1 - 3x_2 = 0$. Letting $x_2 = t$ (an arbitrary scalar), every eigenvector of λ_2 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

So $E_A(-2) = \text{span}([3 \quad 1]^T)$ and $g_A(-2) = 1$.

2. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Also determine the geometric multiplicity of each eigenvalue.

The characteristic equation of A is

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0.$$

Thus the only eigenvalue is $\lambda = 2$. Note that $a_A(2) = 3$. To find the eigenvectors corresponding to $\lambda = 2$, the homogenous linear system represented by $(2I_n - A)\mathbf{x} = \mathbf{0}$ must be solved.

$$2I_3 - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Therefore, using the parameters $s = x_1$ and $t = x_3$ (for arbitrary scalars s and t), the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where s and t are not both zero. So $E_A(2) = \text{span}([1 \quad 0 \quad 0], [0 \quad 0 \quad 1])$, and $g_A(2) = 2$.

Summary - *FINDING EIGENVALUES AND EIGENVECTORS*

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I_n - A| = 0$. It will be a polynomial equation of degree n in the variable λ.

2. Find the roots of the characteristic equation. These are the eigenvalues of A.

3. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I_n - A)\mathbf{x} = \mathbf{0}$. To do this we row-reduce the $n \times n$ matrix $(\lambda_i I_n - A)$. The resulting reduced row-echelon form must have at least one row of zeros (otherwise it would be invertible).

7.1.3 Eigenvalues and Eigenvectors of Linear Transformations

This section began with the definition of eigenvalues and eigenvectors in terms of matrices. However, these definitions could have just as easily been given in terms of linear transformations. A number λ is called an **eigenvalue** of a linear transformation $T : V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the **eigenspace** of λ.

Consider the linear transformation $T : \Re^3 \to \Re^3$, whose matrix relative to the standard basis β is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The matrix of T relative to the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is the diagonal matrix

$$[T]_\gamma^\gamma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

For a given transformation T, how can one know what basis to choose so that the corresponding matrix of transformation is diagonal? Eigenvalues and eigenvectors provide the answer. Consider the matrix of transformation $[T]_\beta^\beta$. Finding the eigenvalues and eigenvectors of

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The eigenvalues can be found by considering the characteristic equation:

$$\begin{aligned}
|\lambda I_3 - A| &= \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} \\
&= (\lambda + 2)[(\lambda - 1)^2 - 9] \\
&= (\lambda + 2)(\lambda^2 - 2\lambda - 8) = (\lambda + 2)^2(\lambda - 4)
\end{aligned}$$

The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenspaces for these two eigenvalues are as follows:

$$E_A(4) = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

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$$E_A(-2) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Two very interesting points should be observed.

- Let $T : \Re^3 \to \Re^3$ be the linear transformation whose standard matrix is A, and let γ be the basis of $\Re^3$ made up by the three linearly independent eigenvectors just found of A. Then $[T]_\gamma^\gamma$, as stated earlier is diagonal

$$[T]_\gamma^\gamma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- The main diagonal entries of the matrix $[T]_\gamma^\gamma$ are the eigenvalues of A.

The next section formalizes these two results and gives the necessary criterion that ensure that a matrix A is similar to a diagonal matrix.

7.2 Diagonalization

The first objective in this section is to show the equivalence (that was suggested at in the previous section) of the following two problems, which on the surface seem quite different.

The Eigenvector Problem: Given an $n \times n$ matrix A, does there exist a basis for $\Re^n$ consisting of eigenvectors of A?

The Diagonalization problem (Matrix Form): Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

Definition

- A square matrix A is called **diagonalizable** if A is similar to a diagonal matrix; that is, if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The matrix P is said to **diagonalize** A.

The following theorem shows that the eigenvector problem and the diagonalization problem are equivalent.

Theorem 7.3. *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

Proof. $(a) \Rightarrow (b)$
Since A is diagonalizable, there is an invertible matrix P

$$
P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}
$$

such that $P^{-1}AP = D$, where

$$
D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}
$$

It follows from the formula $P^{-1}AP = D$ that $AP = PD$; that is,

$$AP = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix}$$

Let $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ denote the column vectors of P. Then the columns of AP are (in order) $\lambda_1 \mathbf{p_1}, \lambda_2 \mathbf{p_2}, \ldots, \lambda_n \mathbf{p_n}$. Considering the product AP by columns, we have:

$$A\mathbf{p_1} = \lambda_1 \mathbf{p_1}, \ A\mathbf{p_2} = \lambda_2 \mathbf{p_2}, \ \ldots, \ A\mathbf{p_n} = \lambda_n$$

Since P is invertible, its column vectors are all nonzero. Thus $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, and $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are the corresponding eigenvectors. Since P is invertible, Theorem 5.10 gives that $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are linearly independent. Thus, A has n linearly independent eigenvectors (and from Theorem 3.5, these form a basis for $\Re^n$).

$(b) \Rightarrow (a)$

Assume that A has n linearly independent eigenvectors, $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

be the matrix whose column vectors are $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$. The column vectors of the product AP are

$$A\mathbf{p_1}, \ A\mathbf{p_2}, \ \ldots, A\mathbf{p_n}$$

But

$$A\mathbf{p_1} = \lambda_1 \mathbf{p_1}, \ A\mathbf{p_2} = \lambda_2 \mathbf{p_2}, \ \ldots, \ A\mathbf{p_n} = \lambda_n \mathbf{p_n}$$

so that

$$AP = \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD$$

where D is the diagonal matrix having the eigenvalues of $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal. Since the column vectors of P are linearly independent, P is invertible; thus $P^{-1}AP = D$ and A is diagonalizable.

□

7.2.1 Procedure for Diagonalizing a Matrix

Theorem 7.3 guarantees than an $n \times n$ matrix with n linearly independent eigenvectors is diagonalizable, and the proof provides the following method for diagonalizing A.

Step 1. Find n linearly independent eigenvectors of A, say, $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$.

Step 2. Form the matrix P having $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ as its column vectors.

Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p_i}$, for $i = 1, 2, \ldots, n$.

In order to carry out Step 1 of this procedure, one first needs a way of determining whether a given $n \times n$ matrix A has n linearly independent eigenvectors, and then one needs a method for finding them. One can address both problems at once by finding bases for the eigenspaces of A. Later in this section it shall be shown that those basis vectors, as a combined set, are linearly independent, so that if there is a total of n such vectors, then A is diagonalizable, and the n basis vectors can be used as the column vectors of the diagonalizing matrix P. If there are fewer than n basis vectors, the A is not diagonalizable.

Examples: *DIAGONALIZING MATRICES*

1. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic equation of A is given by

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the corresponding eigenspaces have the bases shown below

$$\lambda = 2: \quad \mathbf{p_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1: \quad \mathbf{p_3} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix A is diagonalizable and

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. To check this:

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is no preferred order for the columns of P. Since the ith diagonal entry of $P^{-1}AP$ is an eigenvalue of the ith column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. So if we had let

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then the diagonal matrix similar to A would be

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

The characteristic equation of A is given by

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2 = 0$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. The bases for the eigenspaces of λ_1 and λ_2 are:

$$E_A(1) = \mathbf{p_1} = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

$$E_A(2) = \mathbf{p_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable.

There was an assumption made in the previous examples, that the column vectors of P, which are made up of basis vectors from the various eigenspaces of A, are linearly independent. The following theorem addresses this issue.

Theorem 7.4. *If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ then $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is a linearly independent set.*

Proof. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Assume that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly dependent (to obtain a contradiction).

Since an eigenvector is nonzero by definition, $\{\mathbf{v_1}\}$ is linearly independent. Let r be the largest integer such that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is linearly independent. Assuming that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is linearly dependent, r satisfies $1 \leq r < k$. Moreover, by definition of r, $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{r+1}}\}$ is linearly dependent. Thus, there are scalars $c_1, c_2, \ldots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{r+1}\mathbf{v_{r+1}} = \mathbf{0} \tag{7}$$

Multiplying both sides of (7) by A and using

$$A\mathbf{v_1} = \lambda_1\mathbf{v_1}, \ \ A\mathbf{v_2} = \lambda_2\mathbf{v_2}, \ \ \ldots, \ \ A\mathbf{v_{r+1}} = \lambda_{r+1}\mathbf{v_{r+1}}$$

we obtain:

$$c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} + \cdots + c_{r+1}\lambda_{r+1}\mathbf{v_{r+1}} = \mathbf{0} \tag{8}$$

Multiplying both sides of (7) by λ_{r+1} and subtracting the resulting equation from (8) gives

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v_1} + c_2(\lambda_2 - \lambda_{r+1})\mathbf{v_2} + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v_r} = \mathbf{0}$$

Since $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a linearly independent set,

$$c_1(\lambda_1 - \lambda_{r+1}) = c_2(\lambda_2 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

and since $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ are distinct, it follows that

$$c_1 = c_2 = \cdots = c_r = 0 \tag{9}$$

Substituting these values into (7) yields

$$c_{r+1}\mathbf{v_{r+1}} = \mathbf{0}$$

Since the eigenvector $\mathbf{v_{r+1}}$ is nonzero, it follows that

$$c_{r+1} = 0 \tag{10}$$

Equations (9) and (10) contradict the condition that $c_1, c_2, \ldots, c_{r+1}$ are not all zero, which resulted from the assumption that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ were linearly dependent. Thus $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly independent. $\qquad\square$

7.2.2 Conditions on Diagonability

The previous theorem leads to some results regarding the diagonability of matrices. The next theorem is a direct result of Theorem 7.4.

Theorem 7.5. *If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.*

Proof. If $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then by Theorem 7.4 $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are linearly independent. Thus, A is diagonalizable by Theorem 7.3. $\qquad\square$

Examples

1. That the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

   has three distinct eigenvalues, $\lambda_1 = 4, \lambda_2 = 2 + \sqrt{3}, \lambda_3 = 2 - \sqrt{3}$. Therefore, A is diagonalizable. Further,

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 + \sqrt{3} & 0 \\ 0 & 0 & 2 - \sqrt{3} \end{bmatrix}$$

   for some invertible matrix P. If desired, the matrix P can be found using the method outlined previously.

2. The eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

   is a diagonalizable matrix.

There is one final result that is well worth mentioning. Although the proof of this theorem is well beyond the scope of this course even an elementary understanding of this theorem will lead to a fuller understanding of diagonability.

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Theorem 7.6. *If A is a square matrix, then:*

(a) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(b) *A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.*

Given the previous three theorems, we can however easily see that part (b) of Theorem 7.6 follows directly from part (a).

Example

1. Consider the 3×3 matrix
$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$
A is a lower triangular matrix, hence the eigenvalues are the entries on the main diagonal. Therefore $\lambda_1 = 5$ and $\lambda_2 = 4$. Note also that $a_A(5) = 2$ (it occurs on the main diagonal twice), and $a_A(4) = 1$. Consider the eigenspace corresponding to the eigenvalue $\lambda_1 = 5$.
$$E_A(5) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$
Thus $g_A(5) = 1$. Note that $g_A(5) = 1 \leq 2 = a_A(5)$ and hence by Theorem 7.6(b) the matrix A is not diagonalizable. One does not even have to check the dimension of the eigenspace of $\lambda_2 = 4$. Theorem 7.6 becomes extremely handy in matrices of size larger than 3.

7.3 Orthogonal Diagonalization

For most matrices, much of the diagonalization process must be completed before it can be determined whether or not the matrix is diagonalizable. One exception is the set of all triangular matrices with distinct entries on the main diagonal. This section will study another type of matrix that is guaranteed to be diagonalizable: a **symmetric** matrix. It will be shown that these symmetric matrices are diagonalized by a matrix whose columns consist of an orthogonal set of eigenvectors. Before establishing this, however, it is important to grasp some theory regarding **symmetric** and **orthogonal** matrices.

7.3.1 Symmetric Matrices

Definitions

- A square matrix A is **symmetric** if $A = A^T$.

Examples: *SYMMETRIC MATRICES AND NON-SYMMETRIC MATRICES*

1. The following matrices are symmetric:

$$
\begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}
\qquad
\begin{bmatrix} 4 & 3 & 2 \\ 3 & 7 & 6 \\ 2 & 6 & 1 \end{bmatrix}
$$

2. The following matrices are almost symmetric, but almost is not good enough :(
   They are non-symmetric.

$$
\begin{bmatrix} 3 & 4 & 0 \\ 4 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix}
\qquad
\begin{bmatrix} 2 & 5 & 5 \\ 5 & 2 & 8 \\ 6 & 8 & 2 \end{bmatrix}
$$

Unlike non-symmetric matrices, symmetric matrices have a series of very useful properties regarding their eigenvalues which guarantee their diagonability. These properties are summarised in the following theorem.

Theorem 7.7. *If A is an $n \times n$ symmetric matrix, then the following properties are true.*

(a) All eigenvalues of A are real.

(b) If λ is an eigenvalue of A, then $a_A(\lambda) = g_A(\lambda)$ for all λ.

Hence Theorem 7.6 guarantees the diagonability of symmetric matrices.

The proof of this theorem requires some knowledge of complex vector spaces and is therefore omitted. However, the following two examples demonstrate parts (a) and (b) of the theorem respectively.

Examples

1. It can be shown that a 2×2 symmetric matrix always has real eigenvalues and is always diagonalizable. Consider the general 2×2 symmetric matrix A where

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

The characteristic polynomial of A is given by

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned} (a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab - 4c^2 \\ &= a^2 - 2ab + b^2 - 4c^2 \\ &= (a-b)^2 + 4c^2 \end{aligned}$$

Since this discriminant is the sum of two squares, it must be either zero or positive. Therefore the eigenvalues are always real. Also A must always be diagonalizable. Consider the two cases. If $(a-b)^2 - 4c^2 = 0$, then $a = b$ and $c = 0$, which implies that A is already diagonal. That is,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

On the other hand, if $(a-b)^2 + 4c^2 > 0$, then by the Quadratic Formula the characteristic polynomial of A has two distinct real roots, which implies that A has two distinct eigenvalues. Thus A is diagonalizable in this case also.

2. Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

The characteristic polynomial for A is given by

$$|\lambda I_4 - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 & 0 \\ 2 & \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 2 \\ 0 & 0 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2 (\lambda - 3)^2$$

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Thus the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Since each of these eigenvalues has a algebraic multiplicity of 2, from Theorem 7.7(b), the corresponding eigenspaces also have a dimension of 2. Specifically, the eigenspace of $\lambda_1 = -1$ has a basis of

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and the eigenspace of $\lambda_2 = 3$ has a basis of

$$\gamma = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and hence the matrix is diagonalizable.

7.3.2 Orthogonal Matrices

Recall that to diagonalize a square matrix A, an invertible matrix P is needed such that $P^{-1}AP$ is diagonal. For symmetric matrices, it shall be shown that the matrix P can be chosen to have the special property that $P^{-1} = P^T$. This unusual matrix property is defined as follows

Definitions

- A square matrix P is called **orthogonal** if it is invertible and $P^{-1} = P^T$.

Example: *AN ORTHOGONAL MATRIX*

1. The matrix
$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
   is orthogonal since
$$P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Recall also that two vectors $\mathbf{p_1}$ and $\mathbf{p_2}$, are orthogonal in $\Re^n$ if and only if $\mathbf{p_1} \cdot \mathbf{p_2} = 0$, and they are orthonormal if in addition, $\|\mathbf{p_i}\| = 1$ for $i = 1, 2$. In the previous example, it can be observed that the columns of P form an orthonormal set of vectors.

Theorem 7.8. *An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.*

Proof. Suppose that the column vectors of P form an orthonormal set:

$$P = [\mathbf{p_1} \mid \mathbf{p_2} \mid \cdots \mid \mathbf{p_n}]$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

Then the product $P^T P$ has the form

$$P^T P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{p_1} \cdot \mathbf{p_1} & \mathbf{p_1} \cdot \mathbf{p_2} & \cdots & \mathbf{p_1} \cdot \mathbf{p_n} \\ \mathbf{p_2} \cdot \mathbf{p_1} & \mathbf{p_2} \cdot \mathbf{p_2} & \cdots & \mathbf{p_2} \cdot \mathbf{p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p_n} \cdot \mathbf{p_1} & \mathbf{p_n} \cdot \mathbf{p_2} & \cdots & \mathbf{p_n} \cdot \mathbf{p_n} \end{bmatrix}$$

Since the set $\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$ is orthonormal,

$$\mathbf{p_i} \cdot \mathbf{p_j} = 0, \ i \neq j \text{ and } \mathbf{p_i} \cdot \mathbf{p_i} = \|\mathbf{p_i}\|^2 = 1$$

Thus the matrix $P^T P$ has the form

$$P^T P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

Since P is invertible, $P^T = P^{-1}$, and P is orthogonal.

Assume now that P is an $n \times n$ orthogonal matrix. Let P be the matrix

$$P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}$$

As P is orthogonal, $P^T = P^{-1}$ and therefore $P^T P = I_n$.

$$P^T P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{p_1} \cdot \mathbf{p_1} & \mathbf{p_1} \cdot \mathbf{p_2} & \cdots & \mathbf{p_1} \cdot \mathbf{p_n} \\ \mathbf{p_2} \cdot \mathbf{p_1} & \mathbf{p_2} \cdot \mathbf{p_2} & \cdots & \mathbf{p_2} \cdot \mathbf{p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p_n} \cdot \mathbf{p_1} & \mathbf{p_n} \cdot \mathbf{p_2} & \cdots & \mathbf{p_n} \cdot \mathbf{p_n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

where

$$P = [\mathbf{p_1} \mid \mathbf{p_2} \mid \cdots \mid \mathbf{p_n}]$$

therefore one has that

$$\mathbf{p_i} \cdot \mathbf{p_j} = 0, \ i \neq j \text{ and } \mathbf{p_i} \cdot \mathbf{p_i} = \|\mathbf{p_i}\|^2 = 1$$

and hence $\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$ form an orthonormal set. $\qquad\square$

Example: *AN ORTHOGONAL MATRIX*

1. Show that

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

is orthogonal by showing that $P^T P = I_3$.

$$P^T P = [\mathbf{p_1} | \mathbf{p_2} | \mathbf{p_3}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Thus $P^T = P^{-1}$, and P is orthogonal. Moreover it is clear that the columns of P form an orthonormal set, as the following calculations were performed in calculating $P^T P$:

$$\mathbf{p_1} \cdot \mathbf{p_2} = \mathbf{p_1} \cdot \mathbf{p_3} = \mathbf{p_2} \cdot \mathbf{p_3} = 0$$

and
$$\|\mathbf{p_1}\| = \|\mathbf{p_2}\| = \|\mathbf{p_3}\| = 1$$

Before presenting the main result of this section, the following preliminary theorem is given, which states that for symmetric matrices eigenvectors corresponding to distinct eigenvalues are orthogonal.

Theorem 7.9. *Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$ corresponding to λ_1 and λ_2 respectively are orthogonal.*

Proof. Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$. Thus $A\mathbf{x_1} = \lambda_1 \mathbf{x_1}$ and $A\mathbf{x_2} = \lambda_2 \mathbf{x_2}$. To prove the theorem, it is useful to start with the following matrix form of the dot product.

$$\mathbf{x_1} \cdot \mathbf{x_2} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} = \mathbf{x_1}^T \mathbf{x_2}$$

$$\begin{aligned} \lambda_1 (\mathbf{x_1} \cdot \mathbf{x_2}) &= (\lambda_1 \mathbf{x_1}) \cdot \mathbf{x_2} = (A\mathbf{x_1}) \cdot \mathbf{x_2} = (A\mathbf{x_1})^T \mathbf{x_2} \\ &= (\mathbf{x_1}^T A^T)\mathbf{x_2} = (\mathbf{x_1}^T A)\mathbf{x_2} = \mathbf{x_1}^T (A\mathbf{x_2}) \\ &= \mathbf{x_1}^T (\lambda_2 \mathbf{x_2}) = \mathbf{x_1} \cdot (\lambda_2 \mathbf{x_2}) = \lambda_2 (\mathbf{x_1} \cdot \mathbf{x_2}). \end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{x_1} \cdot \mathbf{x_2}) = 0$, and because $\lambda_1 \neq \lambda_2$ it follows that $\mathbf{x_1} \cdot \mathbf{x_2} = 0$. Therefore $\mathbf{x_1}$ and $\mathbf{x_2}$ are orthogonal. $\square$

7.3.3 Orthogonal Diagonalization

Definition

- A matrix A is said to be **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

The following important theorem states that the set of orthogonally diagonalizable matrices is precisely the set of symmetric matrices.

Theorem 7.10. *Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable if and only if A is symmetric.*

Proof. Assume that A is orthogonally diagonalizable. Then there exists an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal. Moreover, since $P^{-1} = P^T$

$$A = PDP^{-1} = PDP^T,$$

which implies that

$$A^T = (PDP^T)^T = (P^T)^T(PD)^T = PD^TP^T = PDP^T = A$$

therefore A is symmetric.

Now assume that A is symmetric. If A has an eigenvalue λ of algebraic multiplicity k, then by Theorem 7.7(b), λ has k linearly independent corresponding eigenvectors. Through the Gram-Schmidt orthonormalization process, this set of k vectors can be used to form an orthonormal basis of eigenvectors for the eigenspace corresponding to λ. This procedure is repeated for each eigenvalue of A. The collection of all resulting eigenvectors is orthogonal by Theorem 7.9, and from the normalization process each vector has magnitude one. Furthermore, these n vectors are linearly independent from Theorem 7.4. Let P be the matrix whose columns consist of these n orthonormal eigenvectors. By Theorem 7.8, P is an orthogonal matrix. Finally by Theorem 7.3, it can be concluded that $P^{-1}AP$ is diagonal. Hence A is orthogonally diagonalizable. $\qquad\square$

The second part of the proof of Theorem 7.10 is *constructive*. That is, it gives steps to follow to obtain an orthogonal matrix P that diagonalizes a symmetric matrix. These steps are summarised below.

7.3.4 Procedure for Orthogonal Diagonalization

Let A be an $n \times n$ symmetric matrix.

Step 1. Find all eigenvalues of A and determine the algebraic multiplicity of each.

Step 2. For each eigenvalue of algebraic multiplicity 1, choose a corresponding eigenvector of magnitude one.

Step 3. For each eigenvalue of algebraic multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. (This is always possible by Theorem 7.7.) If this set is not orthonormal, apply the Gramm-Schmidt orthonormalization process.

Step 4. Steps 2 and 3 produce an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^TAP = D$ will be diagonal. (The main diagonal entries of D are the eigenvalues of A.)

Examples: *ORTHOGONAL DIAGONALIZATION*

1. Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Step 1. The characteristic polynomial of A is

$$
\begin{aligned}
|\lambda I_2 - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} &= (\lambda + 3)(\lambda - 3) - 1^2 \\
&= \lambda^2 - 6\lambda + 9 - 1 \\
&= \lambda^2 - 6\lambda + 8 \\
&= (\lambda - 4)(\lambda - 2)
\end{aligned}
$$

Thus the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$

Step 2. Now for each eigenvalue of algebraic multiplicity 1, an eigenvector is found by solving the homogeneous system $(\lambda I_2 - A)\mathbf{x} = \mathbf{0}$. Note that in this case, both λ_1 and λ_2 have an algebraic multiplicity of 1.

For $\lambda_1 = 4$

$$\lambda_1 I_2 - A = 4I_2 - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \text{rref} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_1 = 4$ is

$$E_A(4) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

For $\lambda_2 = 2$

$$\lambda_2 I_2 - A = 2I_2 - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{rref} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_2 = 2$ is

$$E_A(2) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

The eigenvectors (1,1) and (1,-1) form an *orthogonal* basis for $\Re^2$. Each of these is normalized to produce an *orthonormal* basis. The new bases for the eigenspaces are given below.

$$E_A(4) = \text{span}\{\mathbf{p_1}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \qquad E_A(2) = \text{span}\{\mathbf{p_2}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

Step 3. Because each eigenvalue has an algebraic multiplicity of 1, go directly to step 4.

Step 4. The matrix P is constructed with columns vectors $\mathbf{p_1}$ and $\mathbf{p_2}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2. Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$$

Step 1. The characteristic polynomial of A is

$$\begin{aligned}
|\lambda I_4 - A| &= \begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix} \\
&= (\lambda+7)(\lambda-7)[(\lambda+7)(\lambda-7) - 24^2] \\
&\quad -24^2[(\lambda+7)(\lambda-7) - 24^2] \\
&= [(\lambda+7)(\lambda-7) - 24^2]^2 \\
&= (\lambda^2 - 49 - 576)^2 \\
&= (\lambda^2 - 625)^2 \\
&= [(\lambda+25)(\lambda-25)]^2 \\
&= (\lambda+25)^2(\lambda-25)^2
\end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = -25$. Note also that $a_A(25) = a_A(-25) = 2$.

Step 2. Neither of the eigenvalues have an algebraic multiplicity of one. Go to step 3.

Step 3. The eigenspaces of each eigenvalue are found by solving the homogenous system $(\lambda I_4 - A)\mathbf{x} = \mathbf{0}$. This can be done by converting the matrix $\lambda I_4 - A$ to reduced row-echelon form.

For $\lambda_1 = 25$

$$\lambda_1 I_4 - A = 25I_4 - A = \begin{bmatrix} 32 & -24 & 0 & 0 \\ -24 & 18 & 0 & 0 \\ 0 & 0 & 32 & -24 \\ 0 & 0 & -24 & 18 \end{bmatrix}$$

$$\Longrightarrow \text{rref} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_1 = 25$ is

$$E_A(25) = \text{span}\left\{ \begin{bmatrix} \frac{3}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{3}{4} \\ 1 \end{bmatrix} \right\}$$

For $\lambda_2 = -25$

$$B = \lambda_2 I_4 - A = -25I_4 - A = \begin{bmatrix} -18 & -24 & 0 & 0 \\ -24 & -32 & 0 & 0 \\ 0 & 0 & -18 & -24 \\ 0 & 0 & -24 & -32 \end{bmatrix}$$

$$\Longrightarrow \text{rref}B = \begin{bmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_2 = -25$ is

$$E_A(-25) = \text{span}\left\{ \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix} \right\}$$

These two bases are *orthogonal* but not *orthonormal*. Hence an orthonormal bases for each eigenspace is calculated using the Gramm-Schmidt process. The process yields the following two bases

$$E_A(25) = \text{span}\{\mathbf{p_1}, \mathbf{p_2}\} = \text{span}\left\{ \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \right\}$$

$$E_A(-25) = \text{span}\{\mathbf{p_3}, \mathbf{p_4}\} = \text{span}\left\{ \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \right\}$$

Step 4. Using $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ and $\mathbf{p_4}$ as column vectors, the matrix P is constructed:

$$P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

7.4 Some Applications of Eigenvalues and Eigenvectors

7.4.1 Calculation of Matrix Powers

Let A be an $n \times n$ matrix. To calculate A^m for large m is laborious and time consuming. However if A is diagonalizable this process can be greatly simplified. Consider the following construction.

Let A be an $n \times n$ diagonalizable matrix, and let P be an invertible matrix whose column vectors are the eigenvectors of A, then

$$P^{-1}AP = D$$

where D is a diagonal matrix. If $D = P^{-1}AP$ then it follows that

$$A = PDP^{-1}$$

Therefore $A^m = (PDP^{-1})^m$ where $m \geq 1$. Consider the special cases for $m = 2$ and $m = 3$

$$m = 2 \quad \rightarrow \quad (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$
$$m = 3 \quad \rightarrow \quad (PDP^{-1})^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PDIDIDP^{-1} = PD^3P^{-1}$$

In general
$$A^m = (PDP^{-1})^m = PD^mP^{-1}$$

Recall from Section 1.6.1 that if D is the general diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

then

$$D^m = \begin{bmatrix} d_1{}^m & 0 & \dots & 0 \\ 0 & d_2{}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n{}^m \end{bmatrix}$$

Thus calculating the mth power of a matrix can be done by constructing a matrix P such that $P^{-1}AP = D$ where D is diagonal. Then the calculation A^m, which involves m matrix multiplications, reduces to PD^mP^{-1} which, due to the simplicity of the form of D^m, involves only two matrix multiplications.

Note that this method only works for matrices which are diagonalizable.

Examples: *MATRIX POWERS USING EIGENVALUES AND EIGENVECTORS*

1. Let A be the matrix
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

   The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. The corresponding eigenspaces are spanned by the bases

   $$\{\mathbf{p_1}\} = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \qquad \{\mathbf{p_2}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

   Using $\mathbf{p_1}$ and $\mathbf{p_2}$ as column vectors, the matrix P is constructed:

   $$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

   Then $P^{-1}AP = D$ where

   $$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

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$$\begin{aligned} A^m \;&=\; PD^m P^{-1} \\[2mm] &=\; \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^m \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\[2mm] &=\; \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^m \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\[2mm] &=\; \frac{1}{2} \begin{bmatrix} 3^m + 1 & 3^m - 1 \\ 3^m - 1 & 3^m + 1 \end{bmatrix} \end{aligned}$$

7.4.2   Systems of Linear Differential Equations

A **system of first-order linear differential equations** has the form

$$\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n \\ y_2' &= a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n \\ &\;\;\vdots \\ y_n' &= a_{n1} y_1 + a_{n2} y_2 + \cdots + a_{nn} y_n \end{aligned}$$

where each y_i is a function of t and $y_i' = dy_i/dt$. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

then the system can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}$$

where A is the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Example: *A SIMPLE SYSTEM*

1. Solve the following system of linear differential equations.

$$y_1' = 4y_1$$
$$y_2' = -y_2$$
$$y_3' = 2y_3$$

¿From calculus it is known that the solution of the differential equation

$$y' = ky$$

is

$$y = Ce^{kt}$$

Therefore the solution of the given system is

$$y_1 = C_1 e^{4t}$$
$$y_2 = C_2 e^{-t}$$
$$y_3 = C_3 e^{2t}$$

The matrix form of the system of linear differential equations in the previous example is $\mathbf{y}' = A\mathbf{y}$, or

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Thus the coefficients of t in the solutions $y_i = C_i e^{\lambda_i t}$ are given by the eigenvalues of the matrix A.

If A is a diagonal matrix, then the solution of $\mathbf{y}' = A\mathbf{y}$ can be obtained immediately, as in the previous example. If A is not diagonal, then the solution requires a little more work. First we must find a matrix P that diagonalizes A. Then the change of variables given by $\mathbf{y} = P\mathbf{w}$ and $\mathbf{y}' = P\mathbf{w}'$ produces

$$P\mathbf{w}' = AP\mathbf{w} \quad \Rightarrow \quad \mathbf{w}' = P^{-1}AP\mathbf{w}$$

where $P^{-1}AP$ is a diagonal matrix. This procedure is demonstrated in next example.

Example: *SOLVING BY DIAGONALIZATION*

1. Solve the following system of linear differential equations.

$$y_1' = 3y_1 + 2y_2$$
$$y_2' = 6y_1 - y_2$$

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First it is necessary to find a matrix P which diagonalizes the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 5$, with corresponding eigenvectors $\mathbf{v_1} = (1, -3)$ and $\mathbf{v_2} = (1, 1)$. Since the eigenvalues are distinct, there exists a non-singular matrix P that diagonalizes A. The columns of P are the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. That is,

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

The system represented by $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has the following form.

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} w_1' & = & -3w_1 \\ w_2' & = & 5w_2 \end{matrix}$$

The solution to this system of equations is

$$w_1 = C_1 e^{-3t}$$
$$w_2 = C_2 e^{5t}$$

To return to the original variables y_1 and y_2, use the substitution $\mathbf{y} = P\mathbf{w}$. Then,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$\begin{matrix} y_1 & = & w_1 + w_2 & = & C_1 e^{-3t} + C_2 e^{5t} \\ y_2 & = & -3w_1 + w_2 & = & -3C_1 e^{-3t} + C_2 e^{5t} \end{matrix}$$