Problem Sheet One

1. Which of the following are linear equations in x_1, x_2 and x_3?
   (a) $x_1 + 2x_2 + 7x_3 = 6$ (b) $x_1x_3 + x_2 = 3$
   (c) $x_1 + 3x_3 = -2x_2 + \frac{1}{5}$ (d) $x_1 = 3\sqrt{x_3} + {x_2}^2$
   (e) $x_1 = x_2$ (f) ${x_1}^2 + {x_2}^2 + 2{x_3}^2 = 3^2$

2. For each of the following systems, classify them as non-linear or linear. Further classify those systems which are linear as non-homogenous or homogenous, and form the augmented matrix of the system.

   (a) $$\begin{array}{rcrl} x_1 & - & 2x_2 = & 0 \\ 3x_1 & + & 4x_2 = & -1 \\ 2x_1 & - & x_2 = & 3 \end{array}$$

   (b) $$\begin{array}{rcrcrl} x_1 & - & 3x_2 & + & x_3 = & 0 \\ 5x_1 & - & 2x_2 & - & 3x_3 = & 0 \\ -7x_1 & + & x_2 & + & 2x_3 = & 0 \end{array}$$

   (c) $$\begin{array}{rcrcrl} 2x_1 & - & 3{x_2}^2 & + & x_3 = & 0 \\ -5x_1 & - & 5{x_2}^2 & - & x_3 = & 0 \\ 3x_1 & + & {x_2}^2 & + & x_3 = & 0 \end{array}$$

   (d) $$\begin{array}{rcrcrl} x_1 & & & + & x_3 = & 1 \\ -x_1 & + & 2x_2 & - & x_3 = & 3 \end{array}$$

3. Using elementary row operations, convert the following matrices to reduced row-echelon form.

$$(a) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (d) \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ -4 & 0 & 0 \end{bmatrix}$$

4. Solve the following homogenous system by finding the reduced row-echelon form of the coefficient matrix:

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

5. Solve the following homogenous system by finding the reduced row-echelon form of the coefficient matrix:

$$\begin{array}{rcrcrcrl} -3x_1 & + & x_2 & + & x_3 & + & x_4 = & 0 \\ x_1 & - & 3x_2 & + & x_3 & + & x_4 = & 0 \\ x_1 & + & x_2 & - & 3x_3 & + & x_4 = & 0 \\ x_1 & + & x_2 & + & x_3 & - & 3x_4 = & 0 \end{array}$$

6. Let $A \in M_{n \times n}(\Re)$. Prove the following statements.
   (a) If $A^2 = 0$, A is singular.
   (b) If $A^2 = A$ and $A \neq I_n$, A is singular.

7. Calculate A^9, A^T and A^{-1} when

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Problem Sheet Two

1. Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ belong to a real vector space V. Let
$$U_1 = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m})$$
$$U_2 = \text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$$
Prove that
$$U_1 + U_2 = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$$

2. If **u,v,w** belong to the real vector space V, prove that
$$\text{span}(\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}) = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

3. U and V are subspaces of $\Re^3$ defined by
$$U = \{(x,y,z) \mid x+y+z=0\} \quad \text{and} \quad V = \{(x,y,z) \mid x-y-z=0\}$$
Find spanning families for U and V and prove that $U + V = \Re^3$.

4. Which of the following subsets of $\Re^2$ are subspaces of $\Re^2$?
(a) $\{(x,y) \mid x = 3y\}$ (b) $\{(x,y) \mid x^2 = y^2\}$
(c) $\{(x,y) \mid x + y = 1\}$ (d) $\{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$

5. Which of the following sets of vectors in $\Re^3$ are linearly independent?
(a) (2,-1,2),(3,0,1),(2,2,2)
(b) (3,1,1),(2,-1,5),(1,7,-17)
(c) (6,0,-1),(1,1,4)
(d) (1,3,3),(0,1,4),(5,6,3),(7,2,-1)

6. Which of the following sets of vectors in P_2 are linearly independent?
(a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$
(b) $3 + x + x^2,\ 2 - x + 5x^2,\ 4 - 3x^2$
(c) $6 - x^2,\ 1 + x + 4x^2$
(d) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

7. Let α, β and γ be distinct real numbers. Prove that the vectors $(1, \alpha, \alpha^2)$, $(1, \beta, \beta^2)$ and $(1, \gamma, \gamma^2)$ are linearly independent.

8. Let $u_1, u_2, \ldots, u_n$ be a linearly independent family of vectors in V and let vectors $v_1, v_2, \ldots, v_m \in V$ be defined by
$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq m$$
Prove that $v_1, v_2, \ldots, v_m$ are linearly independent if and only if the rows of the matrix $A = [a_{ij}]$ are linearly independent.

Problem Sheet Three

1. Explain why the following sets of vectors are *not* bases for the indicated vector spaces. (Solve this problem by inspection).

   (a) $\mathbf{u_1} = (1,2), \mathbf{u_2} = (0,3), \mathbf{u_1} = (2,7)$ for $\Re^2$
   (b) $\mathbf{u_1} = (1,2,1), \mathbf{u_2} = (0,3,2)$ for $\Re^3$
   (c) $\mathbf{p_1} = 1 + x + x^2, \mathbf{p_2} = x - 1$ for P_2

2. Which of the following sets of vectors are bases for $\Re^3$?
   (a) (1,0,0),(2,2,0),(3,3,3)
   (b) (3,1,-4),(2,5,6),(1,4,8)
   (c) (2,-3,1),(4,1,1),(0,-7,-1)
   (d) (1,6,4),(2,4,-1),(-1,2,5)

3. Which of the following sets of vectors are bases for P_2?
   (a) $1 - 3x + 2x^2,\ 1 + x + 4x^2,\ 1 - 7x$
   (b) $4 + 6x + x^2,\ -1 + 4x + 2x^2,\ 5 + 2x - x^2$
   (c) $1 + x + x^2,\ x + x^2,\ x^2$
   (d) $-4 + x + 3x^2,\ 6 + 5x + 2x^2,\ 8 + 4x + x^2$

4. Show that the following set of vectors is a basis for $M_{2\times2}(\Re)$.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

In Questions 5 and 6 determine the dimension of and a basis for the solution space of the homogeneous system.

5. $$\begin{aligned} 3x_1 + x_2 + 2x_3 &= 0 \\ 4x_1 \qquad\quad + 5x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

7. Determine bases for the following subspaces of $\Re^3$.
   (a) The plane $3x - 2y + 5z = 0$
   (b) The plane $x - y = 0$
   (c) The line described by the parametric equations

$$\begin{aligned} x &= t \\ y &= -t \qquad -\infty < t < \infty \\ z &= 4t \end{aligned}$$

8. Determine the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

9. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for a vector space V. Show that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is also a basis, where $\mathbf{u_1} = \mathbf{v_1},\ \mathbf{u_2} = \mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{u_3} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$.

10. Find the coordinate vector of $\mathbf{v} = (7, 4)^T$ relative to the basis $(3, 2)^T, (1, 1)^T$ of $\Re^2$.

11. $\mathbf{v_1} = (1, 1, 1)^T, \mathbf{v_2} = (2, 3, 2)^T, \mathbf{v_3} = (1, 5, 4)^T$ form a basis β for $\Re^3$. Vectors $\mathbf{u_1} = (1, 1, 0)^T, \mathbf{u_2} = (1, 2, 0)^T, \mathbf{u_3} = (1, 2, 1)^T$ form a basis γ for $\Re^3$. Find the change of basis matrix $[P]_\beta^\gamma$. Use this matrix to find $[3\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3}]_\gamma$.

12. Find an orthonormal basis for the subspace of $\Re^4$ spanned by

$$\mathbf{u_1} = (1, 1, 1, 1)^T, \ \mathbf{u_2} = (0, 1, 1, 1)^T, \ \mathbf{u_3} = (0, 0, 1, 1)^T$$

Extend this to an orthonormal basis for $\Re^4$.

Problem Sheet Four

1. Find bases for the row space, the column space and the null space of the following matrices. Verify for each matrix that $\dim(R(A)) = \dim(C(A))$ and that $\text{rank}((A)) + \text{nullity}((A)) = n$.

$$\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 0 & 0 & -8 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -2 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -3 & 6 & 6 & 3 \\ 5 & -3 & 10 & 10 & 5 \end{bmatrix}$$

2. Find a basis for the subspace of $\Re^4$ spanned by the given vectors
(a) (1,1,-4,-3),(2,0,2,-2),(2,-1,3,2)
(b) (-1,1,-2,0),(3,3,6,0),(9,0,0,3)
(c) (1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)

3. Find a basis for the subspace of P_2 spanned by the given vectors.
(a) $-1 + x - 2x^2,\ 3 + 3x + 6x^2,\ 9$
(b) $1 + x,\ x^2,\ -2 + 2x^2,\ -3x$
(c) $1 + x - 3x^2,\ 2 + 2x - 6x^2,\ 3 + 3x - 9x^2$

4. Find a basis for the subspace of $M_{2\times 2}(\Re)$ spanned by the vectors

$$\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

5. U and V are subspaces of $\Re^5$ where, $U = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$, $V = \text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$, where $u_1, u_2, u_3, v_1, v_2, v_3$ are the respective columns of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 3 & 4 & 9 & 6 & 8 & 3 \\ -3 & -1 & 0 & 2 & -1 & -1 \\ -1 & -2 & -5 & -2 & -6 & -5 \\ -4 & -2 & -2 & 3 & -5 & -6 \end{bmatrix}$$

Assuming that A has reduced row-echelon form

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find bases for each of the subspaces $U, V, U + V$.

Problem Sheet Five

1. $T : \Re^2 \to \Re^2$ is a linear transformation which maps $(1,2)^T$ to $(-2,3)^T$ and $(1,-1)^T$ to $(5,2)^T$. Find $T(\mathbf{v})$ when $\mathbf{v} = (7,5)^T$.

2. Let U be a vector space with basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. $T : U \to U$ is the linear transformation defined by

$$\begin{aligned} T(\mathbf{u_1}) &= \mathbf{u_1} + \mathbf{u_2} + \mathbf{u_3} \\ T(\mathbf{u_2}) &= \mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3} \\ T(\mathbf{u_3}) &= 2\mathbf{u_1} + 2\mathbf{u_3} \end{aligned}$$

   Find bases for Ker(T), Im(T). Also find rank(T) and nullity(T).

3. Let U be a vector space with basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. $T : U \to U$ is the linear transformation defined by

$$T(\mathbf{u_1}) = \mathbf{u_3} \quad T(\mathbf{u_2}) = -\mathbf{u_3} \quad T(\mathbf{u_3}) = \mathbf{u_1} + \mathbf{u_2}$$

   Find bases for Ker(T), Im(T). Also find rank(T) and nullity(T).

4. Suppose $V = M_{2\times 2}(\Re)$ and $\beta : E_{11}, E_{12}, E_{21}, E_{22}$ is the standard basis for V. Mappings $S, T : V \to V$ are defined by

$$T(A) = \frac{1}{2}(A - A^T), \; S(A) = \frac{1}{2}(A + A^T)$$

   (a) Prove that S and T are linear.
   (b) Find $[S]_\beta^\beta$ and $[T]_\beta^\beta$.
   (c) Find bases for Ker(S) and Im(S), Ker(T) and Im(T).
   (d) Prove that $S^2 = S, T^2 = T$, $ST = 0$ and $TS = 0$.
   (e) Prove that $S + T = I_V$, where $S + T$ is the linear mapping defined by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$.

5. Let $\gamma : \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the standard basis of unit vectors for $V = \Re^3$ and let $\beta : \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be the basis of $\Re^3$ given by

$$\mathbf{v_1} = [1,1,-1]^T, \; \mathbf{v_2} = [2,1,3]^T, \; \mathbf{v_3} = [0,1,1]^T$$

   Find $[I_V]_\beta^\gamma$ and $[I_V]_\gamma^\beta$.

6. Let $T : P_4[\Re] \to P_4[\Re]$ be the linear transformation defined by

$$T(f(x)) = \frac{1}{2}(f(x) + f(-x)).$$

(a) Prove that $T^2 = T$.

(b) For the basis $\beta : 1, x^2, x^4, x, x^3$ of $P_4[\Re]$, find $[T]_\beta^\beta$.

7. Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$ be two bases for $\Re^2$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Find $[T]_\beta^\beta$ and use Theorem 6.12 to calculate $[T]_\gamma^\gamma$ where $T : \Re^2 \to \Re^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_2 \end{bmatrix}$$

Problem Sheet Six

1. For each of the following matrices, determine if they are diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix}$$

2. Evaluate A^m where

   (a)

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

   (b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

3. Solve the following system of linear differential equations by using an appropriate change of variables.

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 + 0x_3 \\ x_2' &= 3x_1 + 0x_2 + -4x_3 \\ x_3' &= 0x_1 + -4x_2 + 2x_3 \end{aligned}$$

4. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

   Find an orthogonal matrix P such that $P^T AP = D$ where D is diagonal.