Example Lecture 14, page 16.

Find the rational canonical form of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}$$

given the characteristic polynomial is $-(t^2+2)^2(t-2)$.

Note $$A^2 = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -12 & 6 \\ 0 & 0 & 4 & -12 & 6 \\ 0 & 0 & 6 & -14 & 6 \\ 0 & 0 & 12 & -24 & 10 \end{pmatrix}$$

$$\Rightarrow A^2 + 2I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -24 & 12 \end{pmatrix}$$

& $$A - 2I = \begin{pmatrix} -2 & 2 & 0 & -6 & 2 \\ 1 & -4 & 0 & 0 & 2 \\ 1 & 0 & -1 & -3 & 2 \\ 1 & -2 & 1 & -3 & 2 \\ 1 & -4 & 3 & -3 & 2 \end{pmatrix}$$

$\Rightarrow (A^2+2I)(A-2I) = 0$.

$\Rightarrow$ minimal polynomial is $(t^2+2)(t-2)$.

subspace K_{t^2+1}:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -24 & 12 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a, b free $\quad c - 2d + e = 0$.

$\Rightarrow$ vectors of the form $\begin{pmatrix} a \\ b \\ 2d-e \\ d \\ e \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + e\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

"T-cyclic" bases:

$$\begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Longrightarrow \beta_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \beta_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix} \right\}.$$

Note

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

&

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow \beta_1 \cup \beta_2$ is a basis for K_{t^2+1}.

Subspace K_{t-2} : (= Eigenspace E_2)

$$\begin{pmatrix} -2 & 2 & 0 & -6 & 2 \\ 1 & -4 & 0 & 0 & 2 \\ 1 & 0 & -1 & -3 & 2 \\ 1 & -2 & 1 & -3 & 2 \\ 1 & -4 & 3 & -3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Row OPERATIONS

$$R_1 \leftrightarrow R_4 \longrightarrow \begin{pmatrix} 1 & -2 & 1 & -3 & 2 \\ 1 & -4 & 0 & 0 & 2 \\ 1 & 0 & -1 & -3 & 2 \\ -2 & 2 & 0 & -6 & 2 \\ 1 & -4 & 3 & -3 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 2R_1 \\ R_5 \to R_5 - R_1 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 1 & -3 & 2 \\ 0 & -2 & -1 & 3 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & -2 & 2 & -12 & 6 \\ 0 & -2 & 2 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - R_2 \\ R_5 \to R_5 - R_2 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 1 & -3 & 2 \\ 0 & -2 & -1 & 3 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & -15 & 6 \\ 0 & 0 & 3 & -3 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ \\ R_4 \to R_4 + R_3 \\ R_5 \to R_5 + R_3 \end{matrix}$$

$$\Longrightarrow \begin{pmatrix} 1 & -2 & 1 & -3 & 2 \\ 0 & -2 & -1 & 3 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow e = 2d, \; c = d, \; b = d, \; a = 0$

$\Longrightarrow \beta_3 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for K_{t-2}.

$\Longrightarrow$ A rational canonical basis is $\beta_1 \cup \beta_2 \cup \beta_3$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\};$$

with corresponding rational canonical form.

$$\left(\begin{array}{cc|cc|c} 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right)$$