Show all your work. Justify your solutions. Answers without justification will not receive full marks.

Only hand in the problems on page 2.

Practice Problems

Question 1. Show that G is abelian iff $aba^{-1}b^{-1} = 1$ for every $a, b \in G$.

Question 2. Are the following subgroups of $GL_n(\mathbb{R})$? Prove your answer.

(a) The set of $n \times n$ real-valued matrices with positive determinant.
(b) The set of $n \times n$ real-valued matrices with determinant -1.

Question 3. Let $\mathscr{M}$ be the set of monotonic functions $\mathbb{R} \to \mathbb{R}$. That is, $f \in \mathscr{M}$ if either $f(x) \geq f(y)\ \forall\ x > y$, or $f(x) \leq f(y)\ \forall\ x > y$. Is $\mathscr{M}$ a subgroup of $\mathscr{F}$ under pointwise addition?

Assignment Problems

Question 1.

(a) Let $G = \mathbb{R}$, $a \diamond b = (a+b)/2$. Is $(K, \diamond)$ a group?

(b) Define an operation $*$ on $H = \mathbb{R} \setminus \{0\}$ by $a * b = |a|b$. Is $(H, *)$ a group?

(c) Define $\odot$ on $K = \mathbb{R} \setminus \{-1\}$ by $a \odot b = ab + a + b$. Is $(K, \odot)$ a group?

Question 2. For $n = 10, 11, 12, 13, 14, 15, 16$, list the elements of the group $(\mathbb{Z}/n\mathbb{Z})^{\times}$. In each case is the group abelian? Is it cyclic? If so, give a generator.

Question 3. Let G be the collection of 2×2 matrices with entries in $\mathbb{Z}/2\mathbb{Z}$ and with determinant 1:

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/2\mathbb{Z}, \quad ad - bc \equiv 1 \pmod 2 \right\}.$$

It is easy to check that G is a group.

(a) List all the elements of G, and calculate their orders.

(b) Is G cyclic? Is it abelian?

Question 4.

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix} \in S_5$. These represent symmetries of a regular pentagon, corresponding to rotation by $2\pi/5$ and a "flip", respectively.

(a) Calculate σ^2, $\sigma^3, \ldots$ until you reach $\sigma^n = 1$. What is the order of σ?

(b) What is the order of τ?

(c) Show that $\tau\sigma = \sigma^4\tau$.

(d) Give the Cayley table for the dihedral group of symmetries of the regular pentagon D_5. Express every element in the table in the form $\sigma^m\tau^n$ where m, $n \geq 0$ are as small as possible.