Show all your work. Justify your solutions. Answers without justification will not receive full marks.

Only hand in the problems on page 2.

Practice Problems

Question 1. Prove that an intersection of normal subgroups of G is a normal subgroup of G.

Question 2. Define operations $\oplus$ and $\odot$ on $\mathbb{Q}$ by $a \oplus b = a + b + 1$, $a \odot b = ab + a + b$. Is $(\mathbb{Q}, \oplus, \odot)$ an integral domain? Is it a field? What happens if $\mathbb{Q}$ is replaced with $\mathbb{Z}$?

Assignment Problems

Question 1.

(a) Let A be the set of real matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and let B be the set of real matrices of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, where a and b are non-zero reals. Prove that $G = A \cup B$ is a group under matrix multiplication.

(b) Prove that A is a subgroup of G, and find the cosets of A in G.

(c) Show that $A \trianglelefteq G$.

(d) Show that the set of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is a normal subgroup of A.

(e) Deduce that "a normal subgroup of a normal subgroup" is not necessarily normal (so normality is not transitive).

Question 2. Let M be the set all 2×2 real matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that $M \simeq \mathbb{C}$.

Question 3. Let $\mathscr{F} = \{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$. Composition of functions $(f, g) \mapsto f \circ g$ is a binary operation on $\mathscr{F}$. Is $(\mathscr{F}, +, \circ)$ a ring?