**DEPARTMENT OF MATHEMATICS**

# MATH2301 Assignment 5 Semester 1, 2010

*Due: May 28, 2pm in the appropriate box in building 67*

Total marks: 50.

**(1)** (12 marks) Let $V$ be a vector space over $\mathbb{R}$ with addition $+$ and scalar multiplication $\cdot$. Let $a$ be a fixed vector in $V$. Define a new addition $\hat{+}$ and scalar multiplication $\hat{\cdot}$ by

$$\begin{aligned} x \hat{+} y &= x + y + a \\ k \hat{\cdot} x &= k \cdot x + (k-1) \cdot a \end{aligned}$$

$\forall x, y \in V$ and $k \in \mathbb{R}$.

Show that $V$ is a vector space under the new operations. Explicitly state the zero vector and the additive inverse of an arbitrary vector $w \in V$.

**(2)** (8 marks) An $n \times n$ square matrix $A$ is called skew-symmetric if $A^T = -A$. Denote the set of $n \times n$ skew-symmetric matrices by $V_n^{\text{S-S}}$, and the set of $n \times n$ symmetric matrices (satisfying $A^T = A$) by $V_n^{\text{S}}$. Let $\mathbb{F}$ be a field that is not of characteristic 2. Show that

$$M_n(\mathbb{F}) = V_n^{\text{S}} \oplus V_n^{\text{S-S}}.$$

Highlight the significance of the assumption that $\mathbb{F}$ is not of characteristic 2.

**(3)** Give a basis and the dimension of each of the following vector spaces:

(a) (3 marks) The space of $n \times n$ matrices with trace $= 0$.

(b) (3 marks) The space of $n \times n$ skew-symmetric matrices.

(c) (3 marks) The space of palindromic vectors in $\mathbb{R}^n$. The following are examples of palindromic vectors:

$$\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \\ b \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \\ c \\ b \\ a \end{pmatrix}, \qquad a, b, c \in \mathbb{R}$$

which are vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^6$ respectively.

**(4)** In the following examples, use the Gram-Schmidt process to give an orthogonal basis for the given inner product spaces.

(a) (5 marks) $P_3(\mathbb{R})$ with basis $\{1, x, x^2, x^3\}$ and inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

If we continued using the Gram-Schmidt process on the basis $\{1, x, x^2, x^3, \dots\}$ of $P(\mathbb{R})$, we would obtain a set of orthogonal polynomials known as the Legendre polynomials. In this question you will be calculating the first four Legendre polynomials. If we label these polynomials as $f_n$ for $n = 0, 1, 2, 3, \dots$, for a given $n$ they satisfy Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

(b) (5 marks) The subspace of $M_2(\mathbb{R})$ with basis

$$\left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$$

and inner product $\langle A, B \rangle = tr(B^T A)$.

**(5)** Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the operator defined by

$$T\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & d \\ a+d & b-c \end{pmatrix}.$$

(a) (2 marks) Show that $T$ is linear.

(b) (3 marks) Give the matrix representation of $T$ with respect to the standard basis of $M_2(\mathbb{R})$, and calculate its eigenvalues and corresponding eigenvectors.

(c) (6 marks) Give bases for the following $T$-invariant subspaces of $M_2(\mathbb{R})$: $ker(T)$, $Im(T)$, all four $T$-cyclic subspaces with respect to the standard basis and any eigenspaces not already in the list.