1. Introduction to vector spaces

The Vector Space $\mathbb{R}^n$

Definition

If n is a positive integer, then an **ordered** n-**tuple** is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called n-**space** and is denoted by $\mathbb{R}^n$.

- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ are called **equal** if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$$

The sum $\mathbf{u}+\mathbf{v}$ is defined by

$$\mathbf{u}+\mathbf{v}=(u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$$

Let k be any scalar, then the **scalar multiple** $k\mathbf{u}$ is defined by

$$k\mathbf{u}=(ku_1, ku_2, \ldots, ku_n)$$

These two operations of addition and scalar multiplication are called the **standard operations** on $\mathbb{R}^n$.
The **zero vector** in $\mathbb{R}^n$ is denoted by **0** and is defined to be the vector

$$\mathbf{0}=(0, 0, \ldots, 0)$$

The **negative** (or **additive inverse**) of u is denoted by -u and is defined by

$$-\mathbf{u}=(-u_1, -u_2, \ldots, -u_n)$$

The **difference** of vectors in $\mathbb{R}^n$ is defined by

$$\mathbf{v}-\mathbf{u}=\mathbf{v}+(-\mathbf{u})$$

If $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in $\mathbb{R}^n$ and k and l are scalars, then:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; that is, $\mathbf{u} - \mathbf{u} = \mathbf{0}$
- $k(l\mathbf{u}) = (kl)\mathbf{u}$
- $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Generalized Vector Spaces

Definition

A *vector space* V over a field $\mathbb{F}$ is a nonempty set on which two operations are defined - addition and scalar multiplication. Addition is a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, and scalar multiplication is a rule for associating with each scalar $k \in \mathbb{F}$ and each object $\mathbf{u}$ in V an object $k\mathbf{u}$ such that

1. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u}+\mathbf{v} \in V$.

2. If $\mathbf{u} \in V$ and $k \in \mathbb{F}$, then $k\mathbf{u} \in V$.

3. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

4. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

5. There is an object $\mathbf{0}$ in V, called a **zero vector** for V, such that $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V.

6. For each $\mathbf{u}$ in V, there is an object -$\mathbf{u}$ in V, called the **additive inverse** of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$;

7. $k(l\mathbf{u}) = (kl)\mathbf{u}$

8. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

9. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

10. $1\mathbf{u} = \mathbf{u}$

Examples of Vector Spaces

The set of all n-tuples with entries in the field $\mathbb{F}$, denoted $\mathbb{F}^n$ (especially note $\mathbb{R}^n$ and $\mathbb{C}^n$).

The set of all $m \times n$ matrices with entries from the field $\mathbb{F}$, denoted $M_{m\times n}(\mathbb{F})$.

Any set of real-valued functions defined on the real line $(-\infty, \infty)$.

The set of polynomials with coefficients from the field $\mathbb{F}$, denoted $P(\mathbb{F})$.

Counter example

Let $V = \mathbb{R}^2$ and define addition and scalar multiplication oparations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0).$$

Some Properties of Vectors

Theorem

If $u, v, w \in V$ (a vector space) such that $u + w = v + w$, then $u = v$.

Corollary

The zero vector and the additive inverse vector (for each vector) are unique.

Theorem

Let V be a vector space over the field $\mathbb{F}$, $\mathbf{u} \in V$, and $k \in \mathbb{F}$. Then the following statements are true:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-k)\mathbf{u} = -(k\mathbf{u}) = k(-\mathbf{u})$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = 0$.

Quiz

True or false?

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $a\mathbf{u} = b\mathbf{u}$ implies $a = b$.

(d) In any vector space, $a\mathbf{u} = a\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$.