10. Eigenvectors and eigenvalues

The diagonalisation problem

Throughout, V is a finite dimensional vector space over a field $\mathbb{F}$.

Given $T \in \ell(V)$, consider:

1. Is there an ordered basis β such that $[T]_\beta$ is diagonal?

2. If there is such a basis, how do we find it?

Definition

$T \in \ell(V)$ is called **diagonalisable** if there is an ordered basis β for V such that $[T]_\beta$ is diagonal. We also call a square matrix A diagonalisable if L_A is diagonalisable.

Discussion

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for V.

- If $[T]_\beta$ is diagonal then $T(v_j) = \lambda_j v_j$.
- Conversely, if $T(v_j) = \lambda_j v_j$ then $[T]_\beta$ is diagonal.

Definition

Let $T \in \ell(V)$. A non-zero vector $v \in V$ is called an **eigenvector** of T if there exists $\lambda \in \mathbb{F}$ such that $T(v) = \lambda v$. We call λ the **eigenvalue** corresponding to v.

Theorem

$T \in \ell(V)$ is diagonalizable if and only if there is an ordered basis β for V consisting of eigenvectors of T.

Theorem

Let $A \in M_n(\mathbb{F})$.Then $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Definition

Let $A \in M_n(\mathbb{F})$. The polynomial

$$f(t) = \det(A - tI_n)$$

is called the **characteristic polynomial** of A.

Lemma

Similar matrices have the same characteristic polynomial.

Definition

Let β be an ordered basis for V, $\dim(V) = n$ and $T \in \ell(V)$. The **characteristic polynomial** $f(t)$ of T is defined to be the characteristic polynomial of $[T]_\beta$.

Example

$T \in \ell(P_2(\mathbb{R}))$ s.t.
$T(f(x)) = f(x) + (x+1)f'(x)$.

Theorem

Let $A \in M_n(\mathbb{F})$.

(a) The characteristic polynomial of A is a polynomial of degree n with leading term $(-1)^n$.

(b) A has no more than n distinct eigenvalues.

Theorem

Let λ be an eigenvalue of $T \in \ell(V)$. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $0 \neq v \in \ker(T - \lambda I)$.

To find eigenvectors of $T \in \ell(V)$ with $\dim(V) = n$, first choose an ordered basis β.

The **standard representation** of V w.r.t. β is the isomorphism

$$\phi_\beta : V \to \mathbb{F}^n \text{ s.t. } \phi_\beta(x) = [x]_\beta.$$

Then $v \in V$ is an eigenvector of T corresponding to λ if and only if $\phi_\beta(v)$ is an eigenvector of $[T]_\beta$.

Quiz
True or false?

- Every linear operator on an n dimensional vector space has n distinct eigenvalues.
- If a real matrix has one eigenvector, then it has infinitely many eigenvectors.
- There exists a square matrix with no eigenvectors.
- Eigenvalues must be non-zero scalars.
- Similar matrices have the same eigenvalues.
- Similar matrices have the same eigenvectors.