11. Diagonalisation

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $T \in \ell(V)$. If $v_1, v_2, \ldots, v_k$ are the corresponding eigenvectors ($v_i \leftrightarrow \lambda_i$), then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Corollary

Let dim$(V) = n$. If $T \in \ell(V)$ has n distinct eigenvalues then T is diagonalisable.

Definition

A polynomial $f(t)$ in $P(\mathbb{F})$ **splits** over $\mathbb{F}$ if there are scalars $c, a_1, \ldots, a_n \in \mathbb{F}$ such that

$$f(t) = c(t - a_1)(t - a_2) \cdots (t - a_n).$$

Theorem

The characteristic polynomial of a diagonalisable linear operator splits.

Definition

Let λ be an eigenvalue of a linear operator with characteristic polynomial $f(t)$. The **(algebraic) multiplicity** of λ is the largest positive integer k such that $(t-\lambda)^k$ is a factor of $f(t)$.

Definition

The **eigenspace** of $T \in \ell(V)$ corresponding to eigenvalue λ is given by

$$E_\lambda = \{x \in V \mid T(x) = \lambda x\} = \mathrm{ker}(T - \lambda I_V).$$

Theorem

Let λ be an eigenvalue of $T \in \ell(V)$ (finite dimensional V) with multiplicity m. Then $1 \leq \mathrm{dim}(E_\lambda) \leq m$.

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $T \in \ell(V)$. For each $i = 1, 2, \ldots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S = S_1 \cup S_2 \cup \ldots \cup S_k$ is a linearly independent subset of V.

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $T \in \ell(V)$ (finite dimensional V) such that the characteristic polynomial of T splits. We have

(a) T is diagonalisable if and only if the multiplicity of λ_i equals $\dim(E_{\lambda_i})$ $\forall i$,

(b) If T is diagonalisable and β_i is an ordered basis for E_{λ_i}, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.

Test for diagonalisation

$T \in \ell(V)$ is diagonalisable if and only if the following hold:

(1) The characteristic polynomial splits

(2) For each eigenvalue λ, the multiplicity of λ equals $\dim(E_\lambda)$.

Example

$T \in \ell(P_2(\mathbb{R}))$ s.t.

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

Example

Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_3(\mathbb{R})$

Example

$T \in \ell(\mathbb{R}^2)$ s.t. $T(a,b) = (3a - b, a + 3b)$.

Quiz
True or false?

- Any linear operator on a n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalisable.

- Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

- If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

- If λ_1 and λ_2 are distinct eigenvalues of $T \in \ell(V)$, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

- Every diagonalisable linear operator on a non-zero vector space has at least one eigenvalue.