12. Cayley-Hamilton theorem

Definition

For $T \in \ell(V)$, a subspace W of V is called T-**invariant** if $T(W) \subseteq W$.

Definition

Let $0 \neq x \in V$. The subspace

$$W = \left\langle x, T(x), T^2(x), \dots \right\rangle$$

is called a T-**cyclic subspace** of V generated by x.

Example

$T \in \ell(\mathbb{R}^3)$ s.t. $T(a, b, c) = (-b + c, a + c, 3c)$

Let W be a T-invariant subspace of V. The restriction of T to W gives rise to a linear operator $T_W \in \ell(W)$.

Theorem

For V f.d., the characteristic polynomial of T_W divides the characteristic polynomial of T.

Example

$T \in \ell(\mathbb{R}^4)$ s.t.
$T(a,b,c,d) = (a+b+2c-d, b+d, 2c-d, c+d)$

Theorem

Let W denote the T-cyclic subspace of V generated by $v \in V$. Also let $k = \dim(W)$. We have

(a) $\left\{v, T(v), T^2(v), \ldots, T^{k-1}(v)\right\}$ is a basis for W.

(b) If

$$a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$$

then the characteristic polynomial of T_W is

$$f(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k).$$

Example

$T \in \ell(\mathbb{R}^3)$ s.t. $T(a,b,c) = (-b+c, a+c, 3c)$

Cayley-Hamilton theorem

Let $T \in \ell(V)$ (f.d. V) and let $f(t)$ be its characteristic polynomial. Then $f(T) = T_0$ (the zero transformation).

Quiz
True or false?

- There exists a linear operator T with no T-invariant subspace.

- Let T be a linear operator on a finite dimensional vector space V and let v and w be in V. If W is the T-cyclic subspace generated by v and W' is the T-cyclic subspace generated by w, and $W = W'$, then $v = w$.

- If T is a linear operator on a finite dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.