13. Jordan canonical form

1. If $T \in \ell(V)$ is not diagonalisable, then at least one eigenspace is too "small".

2. Extend "eigenspace" to "generalised eigenspace".

3. From the "generalised eigenspace", select ordered bases whose union is an ordered basis β for V such that

$$[T]_\beta = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$$

where each "0" is a zero matrix, and each A_i is a square matrix of the form (λ) or

$$A_i = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Definitions

Such a matrix A_i is called a **Jordan block** corresponding to λ and the matrix $[T]_\beta$ is called a **Jordan canonical form** of T.

The associated basis β is referred to as a **Jordan canonical basis**.

Let $A \in M_n(\mathbb{F})$ such that the characteristic polynomial of A splits over $\mathbb{F}$. The **Jordan canonical form** of A is defined to be the Jordan canonical form of L_A.

Main result

If the characteristic polynomial of $T \in \ell(V)$ splits over the underlying field, then T has a Jordan canonical form.

Main problem

Find a Jordan canonical basis.

Example

$T \in \ell(\mathbb{C}^8)$, $\beta = \{v_1, v_2, \ldots, v_8\}$ such that

$$[T]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If v lies in a Jordan canonical basis for a linear operator T and is associated with a Jordan block with diagonal entries λ, then $(T - \lambda I)^p(v) = 0$ for sufficiently large p.

Definition

Let T be a linear operator on a vector space V, and let λ be a scalar. We call $0 \neq x \in V$ a **generalised eigenvector** of T corresponding to λ if $(T - \lambda I)^p(x) = 0$ for some $0 < p \in \mathbb{Z}$.

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The **generalised eigenspace** of T corresponding to λ, denoted K_λ, is the subset of V defined by

$$K_\lambda = \{x \in V \mid (T-\lambda I)^p(x) = 0 \text{ for some } 0 < p \in \mathbb{Z}\}.$$

Definition

Let x be a generalised eigenvector of $T \in \ell(V)$ corresponding to eigenvalue λ. Suppose that p is the smallest positive integer for which $(T - \lambda I)^p(x) = 0$. Then the ordered set

$$\{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \ldots, (T-\lambda I)(x), x\}$$

is called a **cycle of generalised eigenvectors** of T corresponding to λ.

How to form a Jordan canonical basis associated with T?

1. Determine eigenvalues.
2. Find all eigenvectors corresponding to each eigenvalue.
3. For each eigenvector v_i corresponding to a given eigenvalue λ, determine a cycle of generalised eigenvectors.
4. Form a Jordan canonical basis β of V as a disjoint union of cycles.

Theorem

Let A and B be $n \times n$ matrices, each having Jordan canonical forms. Then A and B are similar if and only if they have (up to ordering of eigenvalues) the same Jordan canonical form.

Example

V is the vector space of polynomial functions in two real variables x and y of degree at most 2, with $T \in \ell(V)$ s.t.

$$T(f(x,y)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

$T \in \ell(M_n(\mathbb{R}))$ s.t.

$$T(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \cdot A - A^T.$$