14. Rational canonical form

1. If the characteristic polynomial of $T \in \ell(V)$ does not split, there is no basis on which T has a diagonal form nor a Jordan canonical basis.

2. But the characteristic polynomial factorises as $(-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2}\ldots(\phi_j(t))^{n_j}$.

3. There exists a basis β such that

$$[T]_\beta = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & C_r \end{pmatrix}$$

where each "0" is a zero matrix, and each C_i is a square matrix of the form

$$C_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

Definitions

A polynomial $f(t)$ is called **monic** if its leading coefficient is 1.

If $f(t)$ has positive degree and cannot be expressed as a product of positive-degree polynomials with coefficients in the field, we call $f(t)$ **irreducible**.

A matrix of the form C_i (previous slide) is called a **companion matrix** of the monic polynomial

$$a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k.$$

Use the notation C_x for the T-cyclic subspace generated by $x \in V$. If $\dim(C_x) = k$, then $\{x, T(x), T^2(x) \ldots, T^{k-1}(x)\}$ is a basis for C_x called the T-**cyclic basis generated by** x, denoted β_x. We have that $\exists x \in V$ s.t. $C_i = [T_{C_x}]_{\beta_x}$.

A matrix representation of the form $[T]_\beta$ (from the first slide) is called a **rational canonical form** of T, and the basis β is called a rational canonical basis.

Each C_i in the rational canonical form is a companion matrix of a polynomial $(\phi(t))^m$ such that $\phi(t)$ is an irreducible monic divisor of the characteristic polynomial and $0 < m \in \mathbb{Z}$.

Main result

For every linear operator on a finite dimensional vector space, there exists a rational canonical basis and hence a rational canonical form.

Main problem

Find a rational canonical basis.

Definition

Let T be a linear operator on a finite dimensional vector space V with characteristic polynomial

$$(-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2}\ldots(\phi_k(t))^{n_k}$$

where the $\phi_i(t)$ are distinct irreducible monic polynomials and $0 < n_i \in \mathbb{Z}$. For $1 \leq i \leq k$ we define

$$K_{\phi_i} = \{x \in V \mid (\phi_i(T))^p(x) = 0, \text{ some } 0 < p \in \mathbb{Z}\}.$$

Theorem

β is a rational canonical basis if and only if β is the disjoint union of T-cyclic bases β_{v_i}, where each v_i lies in K_ϕ for some irreducible monic divisor $\phi(t)$ of the characteristic polynomial of T.

Example

$T \in \ell(\mathbb{R}^8)$, $\beta = \{v_1, v_2, \ldots, v_8\}$ such that

$$[T]_\beta = \begin{pmatrix} 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\phi_1(t) = t^2 - t + 3$ and $\phi_2(t) = t^2 + 1$. The diagonal blocks are companion matrices of the polynomials $\phi_1(t)$, $(\phi_2(t))^2$ and $\phi_2(t)$.

Definition

Let $T \in \ell(V)$ for f.d. V. A polynomial $p(t)$ is called a **minimal polynomial** of T if $p(t)$ is a monic polynomial of least degree such that $p(T) = T_0$.

Theorem

Suppose that the minimal polynomial of $T \in \ell(V)$ (f.d. V) is

$$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \dots (\phi_k(t))^{m_k}$$

where the $\phi_i(t)$ are distinct irreducible monic factors of $p(t)$ and $0 < m_i \in \mathbb{Z}$. Then

$$K_{\phi_i} = \ker((\phi_i(T))^{m_i}).$$

If γ_i is a basis for K_{ϕ_i}, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V. Moreover, if each γ_i is a disjoint union of T-cyclic bases, then γ is a rational canonical basis.

How to form a rational canonical basis associated with T?

1. Determine the characteristic polynomial of T.
2. Determine the minimal polynomial of T.
3. Find T-cyclic bases associated to each K_{ϕ_i}.
4. Form a rational canonical basis β of V as a disjoint union of these T-cyclic bases.

Example

$T \in \ell(P_3(\mathbb{R}))$ s.t. $T(f(x)) = f(0)x - f'(1)$.

Let $\beta = \{1, x, x^2, x^3\}$

$$A = [T]_\beta = \begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ characteristic polynomial $(t^2+1)t^2$.

$$\text{Note } A^2 + I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so $(A^2 + I)A = 0$.

The minimal polynomial is therefore

$$(t^2+1)t.$$

To find K_{t^2+1}:

Find all v s.t. $(A^2 + I)v = 0$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a, b$ free and $c, d = 0$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } K_{t^2+1}$$

which is a T-cyclic basis.

To find K_t:

Find all v s.t. $Av = 0$.

$$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a = 0$ and $b = -2c - 3d$

$$\Rightarrow \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } K_t$$

which is the union of two disjoint bases with one vector each (since both are eigenvectors).

A rational canonical basis for T (in coordinate vector form) is

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

which in terms of vectors in $P_3(\mathbb{R})$ is

$$\{1, x, x^2 - 2x, x^3 - 3x\}.$$

The corresponding rational canonical form is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$T \in \ell(M_2(\mathbb{R}))$ s.t. $T(A) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$A = [T]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ characteristic polynomial $(t^2 - t + 1)^2$.

$$\text{Note } A^2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

so $A^2 - A + I = 0$.

The minimal polynomial is therefore

$$t^2 - t + 1.$$

$K_{t^2-t+1} = \mathbb{R}^4$ which has standard basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We seek to turn this into a union of disjoint T-cyclic bases.

Note

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Also note

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$= - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

are two T-cyclic bases whose union gives a rational canonical basis for K_{t^2-t+1}.

A rational canonical basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

with corresponding rational canonical form

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Example

Find the rational canonical form of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}$$

and a corresponding rational canonical basis.

Hint: the characteristic polynomial is

$$-(t^2 + 2)^2(t - 2).$$