2. Subspaces

Definition A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem

If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Examples of Subspaces

- A plane through the origin of $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$.

- A line through the origin of $\mathbb{R}^3$ is also a subspace of $\mathbb{R}^3$.

- Let n be a positive integer, and let $P_n(\mathbb{R})$ denote the set of all functions expressible in the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, \ldots, a_n$ are real numbers. Thus, $P_n(\mathbb{R})$ consists of the zero function together with all real polynomials of degree n or less. The set $P_n(\mathbb{R})$ is a subspace of the vector space of all real-valued functions as well as being a subspace of $P(\mathbb{R})$.

- The *transpose* A^T of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging rows and columns. A *symmetric matrix* is a square matrix A such that $A^T = A$. The set of all symmetric matrices in $M_{n\times n}(\mathbb{F})$ is a subspace of $M_{n\times n}(\mathbb{F})$.

- The trace of an $n \times n$ matrix A, denoted $\mathrm{tr}(A)$, is the sum of the diagonal entries of A. The set of $n \times n$ matrices having trace equal to zero is a subspace of $M_{n\times n}(\mathbb{F})$.

Operations on Vector Spaces

- The addition of two subspaces (of the same vector space) is defined by: $U + V = \{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$

- The intersection $\cap$ of two subsets of a vector space is defined by:

$$U \cap V = \{\mathbf{w} | \mathbf{w} \in U \; and \; \mathbf{w} \in V\}$$

- A vector space W is called the direct sum of U and V, denoted $U \oplus V$, if U and V are subspaces of W with $U \cap V = \{0\}$ and $U + V = W$.

Theorem

Any intersection of subspaces of a vector space V is also a subspace of V.

Quiz
True or false?

(a) If V is a vector space and W is a subset of V that is also a vector space, then W is a subspace of V.

(b) The empty set is a subspace of every vector space.

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(d) The intersection of any two subsets of V is a subspace of V.

(e) Any union of subspaces of a vector space V is a subspace of V.