3. Span and linear dependence

Definitions

- A vector $\mathbf{w}$ is called a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r}$$

  where $k_1, k_2, \ldots, k_r$ are scalars.

- For a set S of vectors in a vector space V, the **span** of S (denoted $span(S)$) is the set consisting of all linear combinations of the vectors in S.

Theorem

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in a vector space V, then:

(a) $span(S)$ is a subspace of V.

(b) $span(S)$ is the smallest subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and every other subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ must contain $span(S)$.

We say that a subset S of a vector space V **spans** V if $span(S) = V$.

Example

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n defined previously since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. This can be denoted by writing

$$P_n = span\{1, x, x^2, \ldots, x^n\}$$

Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r} = \mathbf{0}$$

has at least one solution, namely

$$k_1 = 0, k_2 = 0, \ldots, k_r = 0$$

If this is the only solution, then S is called a **linearly independent** set. If there are other solutions, then S is called a **linearly dependent** set.

Examples

1. If $\mathbf{v_1} = (2, -1, 0, 3), v_2 = (1, 2, 5, -1)$ and $v_3 = (7, -1, 5, 8)$, then the set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent, since $3\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} = \mathbf{0}$.

2. The polynomials

   $\mathbf{p_1} = 1 - x,\ \mathbf{p_2} = 5 + 3x - 2x^2,\ \mathbf{p_3} = 1 + 3x - x^2$

   form a linearly dependent set in P_2 since $3\mathbf{p_1} - \mathbf{p_2} + 2\mathbf{p_3} = \mathbf{0}$

3. Consider the vectors $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$ in $\mathbb{R}^3$. In terms of components the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

becomes

$$k_1(1,0,0)+k_2(0,1,0)+k_3(0,0,1) = (0,0,0)$$

or equivalently,

$$(k_1, k_2, k_3) = (0,0,0)$$

Thus the set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent. A similar argument can be used to extend S to a linear independent set in $\mathbb{R}^n$.

4. In $M_{2\times 3}(\mathbb{R})$, the set

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}$$

is linearly dependent since

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}$$

$$-2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem

Let S_1 and S_2 be subsets of a vector space such that $S_1 \subseteq S_2$.

- If S_1 is linearly dependent, then so is S_2.

- IF S_2 is linearly independent, then so is S_1.

Theorem

Let S be a linearly independent subset of a vector space V, $\mathbf{v} \in V$ and $\mathbf{v} \notin S$. Then $S \cup \{\mathbf{v}\}$ is linearly dependent iff $\mathbf{v} \in span(S)$.

Quiz

True or false?

(a) **0** is a linear combination of any non-empty set of vectors.

(b) If $S \subseteq V$ (vector space V), then $span(S)$ equals the intersection of all subspaces of V that contain S.

(c) If S is a linearly independent set, then each vector in S is a linear combination of other vectors in S.

(d) Any set of vectors containing the zero vector is linearly dependent.