4. Basis and dimension

Definition

If V is any vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a set of vectors in V, then S is called a **basis** for V if the following two conditions hold:

(a) S is linearly independent

(b) S spans V

Theorem

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a subset of a vector space V. Then S is a basis for V iff each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors in S.

Theorem

If a vector space V is spanned by a finite set of vectors S, then S contains a basis for V.

Example:

$$S = \{\, (2,-3,5),\ (8,-12,20),\ (1,0,-2),\ (0,2,-1),$$

$$(7,2,0)\ \}$$

can be shown to span $\mathbb{R}^3$.

Theorem

Let S be a set containing n vectors that spans the vector space V. Let T be a linearly independent set of m vectors in V. We have:
(a) $m \leq n$
(b) $\exists\ U \subseteq S$ containing $n - m$ vectors such that $T \cup U$ spans V.

Corollary

Let the vector space V have a finite basis. Then every basis of V contains the same number of vectors.

Definition

A nonzero vector space V is called **finite dimensional** if it contains a finite set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ that forms a basis. If no such set exists, V is called **infinite dimensional**. The number of vectors in a basis is called the **dimension**. In addition, the zero vector space is regarded as finite dimensional.

Examples

- The vector spaces $\mathbb{F}^n$ and $P_n(\mathbb{F})$ are both finite dimensional.

- The vector space of all real valued functions defined on $(-\infty, \infty)$ is infinite dimensional.

Corollary

Let V be a vector space with dimension n.

(a) Any finite spanning set for V contains at least n vectors, and a spanning set that contains n vectors is a basis.

(b) Any linearly independent subset of V that contains n vectors is a basis.

(c) Every linearly independent subset of V can be extended to a basis.

Theorem

If W is a subspace of a finite dimensional vector space V, then $dim(W) \leq dim(V)$; moreover, if $dim(W) = dim(V)$, then $W = V$.

Corollary

If W is a subspace of a finite dimensional vector space V, then any basis of W can be extended to a basis of V.

Quiz
True or false?

(a) The zero vector space has no basis.

(b) Every vector space that is spanned by a finite set has a basis.

(c) Every vector space has a finite basis.

(d) A vector space cannot have more than one basis.

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(f) Suppose that V is a finite dimensional vector space, S_1 is a linearly independent subset of V, and S_2 is a subset of V that spans V. Then S_1 cannot contain more vectors than S_2.

(g) If S spans the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

(h) Every subspace of a finite dimensional vector space is finite dimensional.

(i) If V is an n dimensional vector space, then V has exactly one subspace with dimension 0 and one with dimension n.

(j) If V is an n dimensional vector space, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.