5. Inner product spaces and orthonormal bases

We only consider $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition

Let V be a vector space over $\mathbb{F}$. We define an inner product $\langle , \rangle$ on V to be a function that assigns a scalar $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{F}$ to every pair of ordered vectors $\mathbf{u}, \mathbf{v} \in V$ such that the following properties hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{F}$:

(a) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(b) $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$

(c) $\overline{\langle \mathbf{u}, \mathbf{v} \rangle} = \langle \mathbf{v}, \mathbf{u} \rangle$

(d) $\langle \mathbf{u}, \mathbf{u} \rangle \ > 0$ if $\mathbf{u} \neq \mathbf{0}$.

Examples

1. $V = \mathbb{F}^n$, $\langle \mathbf{u}, \mathbf{v} \rangle \equiv \mathbf{u} \cdot \mathbf{v}$ determined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i \overline{v_i},$$

$\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.

2. Define $\langle A, B \rangle$ on $M_n(\mathbb{F})$ by

$$\langle A, B \rangle = \text{tr}(B^* A).$$

Theorem

Let V be an inner product space. For $x, y, z \in V$ and $c \in \mathbb{F}$, we have

(a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(d) $\langle x, x \rangle = 0$ iff $x = 0$

(e) If $\langle x, y \rangle = \langle x, z \rangle \; \forall x \in V$, then $y = z$.

Definitions

- A vector space V over $\mathbb{F}$ endowed with a specific inner product is called an inner product space. If $\mathbb{F} = \mathbb{R}$ then V is said to be a real inner product space, whereas if $\mathbb{F} = \mathbb{C}$ we call V a complex inner product space.

- The norm (or length, or magnitude) of a vector $\mathbf{u}$ is given by $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$.

- Two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space are said to be **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors and both $\mathbf{u}$ and $\mathbf{v}$ have a magnitude of one (with respect to $\langle , \rangle$), then $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthonormal**.

- A set of vectors in an inner product space is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has a magnitude of one is called an **orthonormal set**.

Theorem

Every non-zero finite dimensional inner product space V has an orthonormal basis.

Example: Consider the vector space $\mathbb{R}^3$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors $\mathbf{u_1} = (1, 1, 1), \mathbf{u_2} = (0, 1, 1), \mathbf{u_3} = (0, 0, 1)$ into an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$.

Quiz
True or false?

- An inner product is a scalar-valued function on the set of ordered pairs of vectors.

- An inner product space must be over the field of real or complex numbers.

- An inner product is linear in both components.

- If x, y and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.

- If $\langle x, y\rangle = 0$ for all x in an inner product space, then $y = 0$.