6. Linear Transformations

Let V, W be vector spaces over a field $\mathbb{F}$. A function that maps V into W, $T : V \to W$, is called a **linear transformation** from V to W if for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and all scalars $c \in \mathbb{F}$

(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(b) $T(c\mathbf{u}) = cT(\mathbf{u})$

Basic Properties of Linear Transformations

Let $T : V \to W$ be a function.

(a) If T is linear, then $T(\mathbf{0}) = \mathbf{0}$

(b) T is linear if and only if $T(a\mathbf{v}+\mathbf{w}) = aT(\mathbf{v})+T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w}$ in V and $a \in \mathbb{F}$.

In the special case where $V = W$, the linear transformation $T : V \to V$ is called a **linear operator** on V.

Examples

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(a, b) = (2a + b, a)$

2. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ s.t. $T(A) = A^T$

3. $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ s.t.
$T(f(x)) = f'(x)$

4. $C(\mathbb{R})$ is the space of cts real valued functions on $\mathbb{R}$. Fix $a, b \in \mathbb{R}$ s.t. $a < b$. Then

$$T : C(\mathbb{R}) \to \mathbb{R} \text{ s.t. } T(f) = \int_a^b f(t)\ dt.$$

5. *Identity operator:* For any V,
$I : V \to V$ s.t. $I(x) = x$

6. *Zero transformation:* For any V, W,
$T_0 : V \to W$ s.t. $T_0(x) = 0$

Kernel and Image

Definitions

Let $T : V \to W$ be a linear transformation.

The set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T. It is denoted by ker(T). In mathematical notation:

$$\text{ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

The set of all vectors in W that are images under T of at least one vector in V is called the **Image** of T; it is denoted by Im(T). In mathematical notation:

$$\text{Im}(T) = \{\mathbf{w} \in W | \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

Theorem

Let $T : V \to W$ be linear. Then ker(T) and Im(T) are subspaces of V and W respectively.

Example

$T : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $T(a, b, c) = (a - b, 2c)$

Theorem

If $T : V \to W$ is a linear transformation and $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ forms a basis for V, then $\text{Im}(T) = \text{span}(T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots, T(\mathbf{v_n}))$

Rank and Nullity

Definitons If $T : U \to V$ is a linear transformation,

- the dimension of the image of T is called the **rank of** T and is denoted by rank(T),
- the dimension of the kernel is called the **nullity of** T and is denoted by nullity(T).

Example

Let U be a vector space of dimension n, with basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$, and let $T : U \to U$ be a linear operator defined by

$$T(\mathbf{u_i}) = \mathbf{u_{i+1}},\ i = 1, \ldots, n-1,\ T(\mathbf{u_n}) = \mathbf{0}$$

Find bases for ker(T) and Im(T) and determine rank(T) and nullity(T).

Theorem

If $T : V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V) = n$$

Theorem

Let $T : V \to W$ be linear. Then T is injective if and only if $\ker(T) = \{0\}$.

Theorem

Let $T : V \to W$ be linear and $\dim(V) = \dim(W)$. Then the following are equivalent:

- T is injective
- T is surjective
- $\mathrm{rank}(T) = \dim(V)$

Theorem

Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n$ in W there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$, $i = 1, 2, \ldots, n$.

Corollary

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $T_1, T_2 : V \to W$ be linear s.t. $T_1(v_i) = T_2(v_i)$ for $i = 1, 2, \ldots, n$. Then $T_1 = T_2$.

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $T(a, b, c) = (a - b, 2c)$.
Suppose $U : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and

$$U(1,1,1) = (0,2), \quad U(1,0,-1) = (1,-2),$$

$$U(0,-1,1) = (1,-2).$$

Quiz

True or false?

- If $T(x+y) = T(x) + T(y)$ then T is linear.

- If $T : V \to W$ is linear then $T(0_V) = 0_W$.

- T is injective if and only if the only vector x satisfying $T(x) = 0$ is $x = 0$.

- Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ s.t. $T(x_1) = y_1$ and $T(x_2) = y_2$.