7. Matrix representation of a linear transformation

Definition

An **ordered basis** for a finite dimensional vector space V is an basis endowed with a specific order.

Examples

The following bases are called the **standard bases:**

1. $\mathbb{R}^3$...

2. $P_3(\mathbb{R})$...

3. $M_2(\mathbb{R})$...

Definition

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i v_i.$$

The **coordinate vector** of x relative to β is denoted $[x]_\beta$ and is given by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Example

Give the coordinate vector of

$$f(x) = 3 + 2x - x^2$$

with respect to the standard basis of $P_2(\mathbb{R})$.

Let $\beta = \{v_1, v_2, \ldots, v_n\}$, $\gamma = \{w_1, w_2, \ldots, w_m\}$ be bases for vector spaces V and W respectively. For the linear transformation $T : V \to W$, we have

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for each j, $1 \leq j \leq n$. The scalars a_{ij} are unique.

Definition

The $m \times n$ matrix $A = (a_{ij})$ is called the **matrix representation** of T with respect to the bases β and γ, and we write

$$A = [T]_\beta^\gamma.$$

If $V = W$ and $\beta = \gamma$, we simply write

$$A = [T]_\beta.$$

Examples

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$ s.t.
$T(a, b) = (a + 3b, 0, 2a - 4b)$.

2. $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ s.t. $T(f(x)) = f'(x)$.

Definition

Let $T, U : V \to W$ be functions associated with vector spaces V, W over the field $\mathbb{F}$. Define

$$T+U : V \to W, \text{ s.t. } (T+U)(x) = T(x)+U(x),$$

$$kT : V \to W, \text{ s.t. } (kT)(x) = kT(x).$$

Theorem

Let V and W be vector spaces over $\mathbb{F}$, and let $T, U : V \to W$ be linear.

(a) For all $k \in \mathbb{F}$, $kT + U$ is linear.

(b) The collection of all linear transformations from V to W is a vector space, denoted $\ell(V, W)$.

Theorem

Let V and W be finite-dimensional vector spaces with ordered bases β and γ respectively. Let $T, U : V \to W$ be linear transformations. Then

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[kT]_\beta^\gamma = k[T]_\beta^\gamma, \ \forall k$

Quiz

True or false?

Let V and W be finite dimensional vector space with ordered bases β and γ respectively. $T, U : V \to W$ are linear.

- For any scalar k, $kT + U$ is linear.
- $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies $T = U$.
- If m =dim(V) and n =dim(W), then $[T]_\beta^\gamma$ is an $m \times n$ matrix.
- $\ell(V, W)$ is a vector space.