8. Isomorphisms

Definition

Let $T : V \to W$ be linear. A function $U : W \to V$ is called an **inverse** of T if $TU = I_W$ and $UT = I_V$. If T has an inverse then it is called **invertible**.

- The inverse of T is unique and denoted T^{-1}
- T is invertible iff T is bijective
- T^{-1} is linear

Example

$T : P_1(\mathbb{R}) \to \mathbb{R}^2$ s.t. $T(a + bx) = (a, a + b)$

Lemma

Let $T : V \to W$ be linear and invertible. Then V is finite dimensional if and only if W is finite dimensional, with $\dim(V) = \dim(W)$.

Theorem

Let $T : V \to W$ be linear, with finite dimensional vector spaces V and W with bases β and γ respectively. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Moreover, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Example

Let A be an $m \times n$ matrix. Let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be defined by $L_A(x) = Ax \ \forall x \in V$. We call L_A the **left multiplication transformation**.

Example

$T : P_1(\mathbb{R}) \to \mathbb{R}^2$ s.t. $T(a + bx) = (a, a + b)$

Definition

Let V and W be vector spaces. We say that V is **isomorphic** to W if there is an invertible linear transformation $T : V \to W$. We call such a T an **isomorphism**.

Examples

1. $T : \mathbb{R}^4 \to P_3(\mathbb{R})$ s.t. $T(a, b, c, d) = a + bx + cx^2 + dx^3$.

2. $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$
   s.t. $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3$

Theorem

Let V and W be finite dimensional vector spaces over the field $\mathbb{F}$. Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Corollary

A vector space V over the field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ if and only if $\dim(V) = n$.

Theorem

Let V and W be finite dimensional vector spaces over $\mathbb{F}$ with dimensions n and m. Let β and γ be ordered bases for V and W respectively. Then the function
$\Phi : \ell(V, W) \to M_{m\times n}(\mathbb{F})$ s.t. $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Quiz
True or false?

- T is invertible if and only if T is injective and surjective.

- $T : M_2(\mathbb{R}) \to P_2(\mathbb{R})$
s.t. $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + (b+c)x + (c+d)x^2$
is invertible.

- $M_{2\times 3}(\mathbb{R})$ is isomorphic to $\mathbb{R}^5$.

- $P_n(\mathbb{R})$ is isomorphic to $P_m(\mathbb{R})$ if and only if $m = n$.

- $T : \mathbb{R}^3 \to \mathbb{R}^3$ s.t.
$T(a,b,c) = (a+b, b+c, a-c)$ is an isomorphism.