9. Change of basis/coordinates

Theorem

Let β and β' be two ordered bases for a finite dimensional vector space V. We have:

(a) $[I_V]_\beta^{\beta'}$ is invertible

(b) For any $x \in V$, $[x]_{\beta'} = [I_V]_\beta^{\beta'}[x]_\beta$.

Definition

The matrix $[I_V]_\beta^{\beta'}$ is called a **change of coordinate matrix**. It is said to change β-coordinates into β'-coordinates.

Example

In $\mathbb{R}^2$, let $\beta' = \{(1,1),(1,-1)\}$ and $\beta = \{(2,4),(3,1)\}$.

Theorem

Let V be finite dimensional and let $T \in \ell(V)$. Let β and β' be ordered bases for V. We have

$$[T]_{\beta'} = [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta}.$$

Example

$T \in \ell(\mathbb{R}^2)$ s.t. $T(a,b) = (3a - b, a + 3b)$.

Corollary

Let $A \in M_n(\mathbb{F})$, and let β be an ordered basis for $\mathbb{F}^n$. Then $[L_A]_\beta = P^{-1}AP$, where the jth column of $P \in M_n(\mathbb{F})$ is the jth vector of β.

Example

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}, \beta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Definition

Let $A, B \in M_n(\mathbb{F})$. The matrix B is said to be **similar** to A if there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

Assume the following:
- V and W are finite dimensional vector spaces
- $T : V \to W$ is linear
- β and β' are ordered bases for V
- γ and γ' are ordered bases for W

Then we have

$$[T]_{\beta'}^{\gamma'}[I_V]_{\beta}^{\beta'} = [I_W]_{\gamma}^{\gamma'}[T]_{\beta}^{\gamma}$$

Quiz

True or false?

- Every change of coordinate matrix is invertible.

- Let T be a linear operator on a finite dimensional vector space V, let β and β' be ordered bases for V and let P be the change of coordinates from β' to β. Then $[T]_\beta = P[T]_{\beta'}P^{-1}$.

- Let T be a linear operator on a finite dimensional vector space V. Then for any ordered bases β and γ of V, $[T]_\beta$ is similar to $[T]_\gamma$.