Math 2400
Assignment 1 - Solutions

1. The set $\mathbb{F} = \{0, 1, \alpha\}$ forms a field when equipped with the operations $+$ and $\cdot$ defined by:

$+$	0	1	α
0	0	1	α
1	1	α	0
α	α	0	1

$\cdot$	0	1	α
0	0	0	0
1	0	1	α
α	0	α	1

Clearly 0 is an additive identity and each element has an additive inverse since 0 occurs in every row and column, symmetrically about the leading diagonal. In fact this symmetry applies to the entire Cayley table for $+$, so this operation is commutative and $(\mathbb{F}, +)$ forms an abelian group. Similar considerations show that $(\mathbb{F} \setminus \{0\}, \cdot)$ is also an abelian group. Direct computations (you needed to do these for full marks) show that both operations are associative and that the distributive laws hold.

2. Let p be the number of elements in $\mathbb{F}$ and define $\sigma_l \in \mathbb{F}$ by

$$\sigma_l = \underbrace{1 + 1 + \cdots + 1}_{l \text{ times}}.$$

The set $\{\sigma_k : 1 \leq k \leq p+1\}$ is a subset of $\mathbb{F}$ that contains $p+1$ elements, so at least two of these are equal. That is, there must be natural numbers l and m satisfying $1 \leq l < m \leq p+1$ such that $\sigma_l = \sigma_m$, in which case

$$\begin{aligned} 0 &= \sigma_m - \sigma_l \\ &= \underbrace{1 + 1 + \cdots + 1}_{m \text{ times}} - \underbrace{1 + 1 + \cdots + 1}_{l \text{ times}} \\ &= \underbrace{1 + 1 + \cdots + 1.}_{m - l \text{ times}} \end{aligned}$$

This is the result desired, with $n = m - l$.

3. Suppose that $x^2 = 6$ and $x = \frac{p}{q}$, where p and $q \neq 0$ are integers with no divisors in common. Then we have $6q^2 = p^2$, which implies that p^2 is even. If p is odd then there is some integer k for which $p = 2k + 1$. However this implies that

$$p^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is also odd, so p must be even. We know then that $p = 2l$ for some integer l, so $3q^2 = 2l^2$ which is even. If q is odd, of the form $2m + 1$ say, then

$$3q^2 = 3(2(2m^2 + 2m) + 1) = 2(6m^2 + 6m + 1) + 1$$

which is odd. Therefore p and q are both even, but we assumed that they had no common divisors, so this is a contradiction.

4. By the usual triangle inequality, for all $x, y \in \mathbb{R}$ we have

$$|x| - |y| = |x - y + y| - |y| \leq |x - y| + |y| - |y| = |x - y|$$

and

$$|x| - |y| = |x| - |-x + x - y| \geq |x| - |x| - |x - y| = -|x - y|.$$

Therefore,

$$-|x-y| \leq |x| - |y| \leq |x-y|$$

which is equivalent to

$$||x| - |y|| \leq |x-y|.$$

5. For $n \geq 1$ it holds that $n(n-1) \geq 0 \Rightarrow n^2 \geq n$, so

$$\frac{1}{n^2+n} \leq \frac{1}{2n}.$$

Now, given $\epsilon > 0$, whenever

$$\frac{1}{2\epsilon} < n$$

we have

$$\frac{1}{2n} < \epsilon.$$

Clearly then it suffices to choose $N = \left\lceil \frac{1}{2\epsilon} \right\rceil$ to ensure that

$$\left| \frac{1}{n^2+n} \right| < \epsilon$$

for all $n \geq N$. Since ϵ was arbitrary we may conclude that

$$\lim_{n\to\infty} \frac{1}{n^2+n} = 0.$$

6. If $(a_n)_{n=1}^{\infty}$ converges it is a Cauchy sequence, so there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{1}{2}$$

for all $n, m \geq N$. However if $b, c \in \mathbb{Z}$ are distinct then

$$|b-c| \geq 1$$

so it must be the case that $a_m = a_n$ for all $n, m \geq N$. That is, it is necessary that after finitely many terms the sequence becomes constant. This is also a sufficient condition for convergence - if there exists $N \in \mathbb{N}$ such that $a_n = a_m$ for all $n, m \geq N$ then $|a_n - a_m| = 0$, which is less than every $\epsilon > 0$, whenever $n, m \geq N$.