Math 2400
Assignment 2 - Solutions

1. The map that sends a triple (p,q,r) to the integer $2^p 3^q 5^r$ is an injection. This is a consequence of the fundamental theorem of arithmetic - every integer admits a unique factorisation into a product of powers of prime numbers so if $2^{p_1}3^{q_1}5^{r_1} = 2^{p_2}3^{q_2}5^{r_2}$ then $(p_1,q_1,r_1) = (p_2,q_2,r_2)$.

2. Let
$$f_n(x) = \lim_{k\to\infty} (\cos n!\pi x)^{2k}.$$
If $x \in \mathbb{Q}$, say $x = p/q$ where $p,q \in \mathbb{Z}$ and $q \neq 0$, then for $n \geq q$,
$$\begin{aligned} n!x &= 1 \cdot 2 \cdot \cdots \cdot (q-1) \cdot q \cdot (q+1) \cdot \cdots \cdot (n-1) \cdot n \cdot \frac{p}{q} \\ &= 1 \cdot 2 \cdot \cdots \cdot (q-1) \cdot (q+1) \cdot \cdots \cdot (n-1) \cdot n \cdot p \end{aligned}$$
which is an integer. Therefore
$$\cos n!\pi x = \begin{cases} 1 & \text{if } n!x \text{ is even} \\ -1 & \text{if } n!x \text{ is odd,} \end{cases}$$
so we know that whenever $n \geq q$, $f_n(x) = 1$. Consequently,
$$\lim_{n\to\infty} f_n(x) = 1.$$
Otherwise if x is irrational, $n!x \notin \mathbb{Z}$ so $|\cos n!\pi x| < 1$ and for any fixed n the geometric progression $(\cos n!\pi x)^{2k} \to 0$ as $k \to \infty$. Therefore $f_n(x) = 0$ for all n and
$$\lim_{n\to\infty} f_n(x) = 0.$$

3. Let $\lim_{n\to\infty} a_n = a$ and fix $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|a_n - a| < \epsilon$. Since $a_n \in [0,1]$ for all n we know that $|a_n - 1/2| \leq 1/2$, which implies
$$\begin{aligned} |a - 1/2| &= |a - a_n + a_n - 1/2| \\ &\leq |a_n - a| + |a_n - 1/2| \\ &< \epsilon + 1/2. \end{aligned}$$
Since ϵ was arbitrary, it must be the case that $|a - 1/2| \leq 1/2$, or in other words $a \in [0,1]$.

4. (a) Since $(x_n)_{n=1}^\infty$ is bounded we may define
$$\begin{aligned} S_n &= \sup\{x_i : i \geq n\} \\ I_n &= \inf\{x_i : i \geq n\}. \end{aligned}$$
Furthermore
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} I_n$$
both exist, which implies that
$$\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n = \lim_{n\to\infty} (S_n - I_n).$$
For all $m \geq n$ we have $I_n \leq x_m \leq S_n$, so $S_n - I_n \geq 0$ for all n and we conclude (using the same technique employed in question 3) that
$$\lim_{n\to\infty} (S_n - I_n) \geq 0 \;\Rightarrow\; \limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n.$$

(b) Set $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$. Then since $1 + \frac{1}{n}$ is monotone decreasing in n, using the definitions above we have:

$$S_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is even} \\ 1 + \frac{1}{n+1} & \text{if } n \text{ is odd} \end{cases}$$

$$I_n = \begin{cases} -1 - \frac{1}{n+1} & \text{if } n \text{ is even} \\ -1 - \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Now $|S_n - 1| \leq \frac{1}{n}$, so given $\epsilon > 0$ it suffices to choose $N = \left\lceil \frac{1}{\epsilon} \right\rceil$ to ensure that $|S_n - 1| < \epsilon$ whenever $n \geq N$. Therefore

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n = 1.$$

Similarly,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} I_n = -1.$$

5. (a) Let $P_n = \prod_{k=1}^n b_k$. The sequence P_n converges to $p \neq 0$ so there exists an $N_1 \in \mathbb{N}$ and $\alpha > 0$ such that $|P_n| \geq \alpha$ for all $n \geq N_1$ (for example $\alpha = |p/2|$ would work).

   Convergence also implies the Cauchy condition - let $\epsilon > 0$ be arbitrary and fix $N_2 \in \mathbb{N}$ such that $|P_n - P_m| < \alpha\epsilon$ whenever $n, m \geq N_2$. Then if $n \geq N = \max\{N_1, N_2\}$ we have

$$\begin{aligned} \alpha|b_{n+1} - 1| &\leq |P_n||b_{n+1} - 1| \\ &= \left|\prod_{k=1}^n b_k\right| |b_{n+1} - 1| \\ &= \left|\prod_{k=1}^{n+1} b_k - \prod_{k=1}^n b_k\right| \\ &= |P_{n+1} - P_n| \\ &< \alpha\epsilon. \end{aligned}$$

   Therefore $|b_n - 1| < \epsilon$ for all $n \geq N + 1$, and we have

$$\lim_{n\to\infty} b_n = 1.$$

   (b) Let $b_k = \frac{k^3+k^2+k}{k^3+1} = \frac{k(k^2+k+1)}{(k+1)(k^2-k+1)}$ and $P_n = \prod_{k=1}^n b_k$. Writing out the first few values of P_n makes it clear that the product is telescoping:

$$\begin{aligned} P_1 &= \frac{3}{2} \\ P_2 &= \frac{3}{2} \cdot \frac{2 \cdot 7}{3 \cdot 3} = \frac{7}{3} \\ P_3 &= \frac{3}{2} \cdot \frac{2 \cdot 7}{3 \cdot 3} \cdot \frac{3 \cdot 13}{4 \cdot 7} = \frac{13}{4}. \end{aligned}$$

   This leads to the conjecture that

$$P_n = \frac{n^2+n+1}{n+1}. \tag{1}$$

   We prove this by induction:

Clearly (1) holds when $n = 1$. If $P_k = \frac{k^2+k+1}{k+1}$ then

$$\begin{aligned}
P_{k+1} &= P_k \cdot b_{k+1} \\
&= P_k \cdot \frac{(k+1)((k+1)^2 + (k+1) + 1)}{(k+2)((k+1)^2 - (k+1) + 1)} \\
&= \frac{k^2+k+1}{k+1} \cdot \frac{(k+1)((k+1)^2 + (k+1) + 1)}{(k+2)(k^2+k+1)} \\
&= \frac{(k+1)^2 + (k+1) + 1}{k+2},
\end{aligned}$$

so (1) holds for $n = k+1$ as well.

Now note that $n^2 + n + 1 \geq n^2 + n = n(n+1)$, so for $n \geq 1$ we have

$$P_n = \frac{n^2+n+1}{n+1} \geq n,$$

and by comparison $\lim_{n\to\infty} P_n$ does not exist. Therefore $\prod_{k=1}^{\infty} b_k$ does not converge.

Note however that $\lim_{k\to\infty} b_k = 1$, so in part (a) we proved a necessary but not sufficient condition for the convergence of infinite products.