MATH2400 Assignment 3 Solutions

30th April 2014

1 Question 1

1.1 Question 1(a)

The function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = e^{-x}$$

is not uniformly continuous (on $\mathbb{R}$).

Proof. Set $\varepsilon \equiv 1$ and let $\delta > 0$ be given.

Let the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be defined as

$$\begin{aligned} x_n &\equiv \log(n+1) \text{ and} \\ y_n &\equiv \log(n), \end{aligned}$$

respectively. Then

$$|x_n - y_n| = \log\left(\frac{n+1}{n}\right) = \log\left(1 + \frac{1}{n}\right) \quad \forall\, n \in \mathbb{N}.$$

Since $\log(1 + 1/n) \to 0$ as $n \to \infty$, there exists an $N \in \mathbb{N}$ such that for each $n > N$ we have $\log(1 + 1/n) < \delta$ and consequently

$$|x_n - y_n| < \delta \quad \forall\, n > N,$$

but

$$|e^{-x_n} - e^{-y_n}| = |-n - 1 + n| = 1 \geq \varepsilon.$$

Thus, f is not uniformly continuous on $\mathbb{R}$. □

1.2 Question 1(b)

The function $f : [0, \infty) \to \mathbb{R}$ defined as

$$f(x) \equiv e^{-x^2}$$

is uniformly continuous $\big($on $[0, \infty)\big)$.

Proof. Note that the function is a continuous function that is (strictly) monotonically decreasing to zero. It is a continuous function because it is a composition of continuous functions. It is monotonically decreasing because:

$$x_2 > x_1 \geq 0 \Rightarrow x_2^2 > x_1^2 \Rightarrow e^{-x_2^2} < e^{-x_1^2}.$$

For all $x \in [0, \infty)$ we have $f(x) > 0$ and since f is strictly monotonically decreasing we have

$$\lim_{x \to \infty} f(x) = 0. \tag{1}$$

Given $\varepsilon > 0$ we wish to show that there exists $\delta = \delta(\varepsilon) > 0$ such that if for all $x, y \geq 0$ we have $|x - y| < \delta$ then

$$|f(x) - f(y)| < \varepsilon.$$

Furthermore, note that $0 < f \leq 1$ so wlog (without loss of generality) we need not consider the case that $\varepsilon \geq 1$ as uniform continuity follows for any $\delta > 0$ in this case.

Note that $f\big([0, \infty)\big) = (0, 1]$, i.e. f is surjective. One can prove surjectivity as an application of the IVT. We know that $f(0) = 1$ so given $y \in (0, 1)$ (1) implies there exists an $x_1 > 0$ such that

$$f(x_1) < \frac{y}{2}$$

We know the function is continuous on $[0, x_1]$ with $f(0) > y > y/2 > f(x_1)$. Consequently, by the IVT, there exists an $x_0 \in (0, x_1)$ such that $f(x_0) = y$. The monotonicity property implies it is injective. Thus $f : [0, \infty) \to (0, 1]$ is bijective.

Let $1 > \varepsilon > 0$ be given. By bijectivity of f there exists a unique $x_0 > 0$ such that $f(x_0) = \varepsilon/2$.

The function f is continuous on $[0, x_0]$ and therefore uniformly continuous on $[0, x_0]$ (this result should have been proved in class). Consequently, there exists a $\delta^* > 0$ such that for all $x, y \in [0, x_0]$ we have

$$|x - y| < \delta^* \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}. \tag{2}$$

Since $f \downarrow 0$, for any $x, y \geq x_0$ the RHS of (2) is always true, so it suffices to choose δ^*.

So now consider the case that I is an interval of the form (a, b) or $[a, b)$ or $(a, b]$ such that $0 \leq a < x_0,\ b > x_0$ and $b - a < \delta^*$. Then for all $x, y \in I$ we have $|x - y| < \delta^*$ and this implies via (2) that

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus having considered all possible cases we have shown that if an arbitrary interval $(x, y) \subset [0, \infty)$ is such that $|x - y| < \delta^*$ then we have that

$$|f(x) - f(y)| < \varepsilon.$$

This argument can be replicated for any $\varepsilon > 0$ so f is uniformly continuous on $[0, \infty)$.

$\square$

2 Question 2

We wish to show that there exists a $c \in \mathbb{R}$ such that

$$2\left(\frac{2+|c|}{1+|c|}\right)^{1+|c|} = 5$$

Proof. Consider the function $f : [0, 100] \to \mathbb{R}$ defined as

$$f(x) \equiv \left(\frac{2+|x|}{1+|x|}\right)^{1+|x|}.$$

Then f is a composition of continuous functions and hence continuous itself. Moreoever, we have

$$2 = f(0) < \frac{5}{2} < f(100) = \left(\frac{102}{101}\right)^{101}.$$

Thus, by the intermediate value theorem, there exists a $c \in (0, 100)$ such that

$$f(c) = \frac{5}{2}$$

which is true iff.

$$2f(c) = 5.$$

□

3 Question 3

3.1 Question 3(a)

We want to show:

$$f \in C(\mathbb{R}; \mathbb{R}) \Rightarrow |f| \in C(\mathbb{R}; \mathbb{R})$$

Proof. Given $x \in \mathbb{R}$ let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence such that $x_n \to x$ as $n \to \infty$, then by the reverse triangle inequality and the continuity of f at x we have

$$\big|\, |f(x_n)| - |f(x)| \,\big| \leq |f(x_n) - f(x)| \to 0 \quad \text{as } (n \to \infty).$$

That is

$$x_n \to x \Rightarrow |f(x_n)| \to |f(x)|.$$

This argument is true for any $x \in \mathbb{R}$ whence we conclude $|f|$ is continuous on $\mathbb{R}$.

□

3.2 Question 3(b)

The function $f : \mathbb{R} \to \mathbb{R}$ given below:

$$f(x) \equiv \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

is discontinuous everywhere but $|f(x)| \equiv 1$ is continuous on $\mathbb{R}$. We give the proof below for reference only, that is, it is not required as part of the solution.

Proof. Constant functions are clearly continuous. We will show that f is actually discontinuous everywhere. If $x \in \mathbb{Q}$ then

$$x_n \equiv x + \frac{\pi}{n} \in \mathbb{R} \setminus \mathbb{Q} \quad \forall\, n \in \mathbb{N}$$

and as $n \to \infty$ we have $x_n \to x$ but

$$f(x_n) = 1 \to 1 \neq -1 = f(x)$$

so f is discontinuous on $\mathbb{Q}$. If $x \in \mathbb{R} \setminus \mathbb{Q}$ then $x \in [N, N+1]$ for some unique $N \in \mathbb{Z}$. Given $n \in \mathbb{N}$ we can write $[N, N+1]$ as $\cup_{j=1}^{n+1} [N + \frac{j}{n+1}] = \bigcup_{j=1}^{n+1} I_j$ and construct a sequence of rational numbers that get arbitrarily close to x as follows:

$$a_n \equiv N + \frac{j}{n+1}$$

where j is an element in $\{1, 2, 3, \ldots, n+1\}$ such that

$$\left| N + \frac{j}{n+1} - x \right| = \min_{k \in \{1,2,\ldots,n+1\}} \left| N + \frac{k}{n+1} - x \right|.$$

Then for each $n \in \mathbb{N}$ we have $a_n \in \mathbb{Q}$ and $|a_n - x| \to 0$ as $n \to \infty$ by the above construction but

$$-1 = f(a_n) \to -1 \neq 1 = f(x)$$

Thus, f is also discontinuous on $\mathbb{R} \setminus \mathbb{Q}$, and consequently, f is discontinuous everywhere. □

4 Question 4

We want to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and satisfies

$$f(x+y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R} \tag{3}$$

then f is continuous on all of $\mathbb{R}$.

Proof. Firstly, for any $x, y \in \mathbb{R}$, (3) implies

$$f(x) = f(x - y + y) = f(x-y) + f(y),$$

and therefore

$$f(x) - f(y) = f(x - y) \quad \forall\, x, y \in \mathbb{R}. \tag{4}$$

In particular, for $x = y$ we have $f(0) = 0$.

Now, given any $x \in \mathbb{R}$, let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty}(x_n - x) = 0$. Then

$$\begin{aligned} \lim_{n\to\infty} f(x_n) - f(x) &= \lim_{n\to\infty} \big(f(x_n) - f(x)\big) \\ &= \lim_{n\to\infty} f(x_n - x) \quad \text{via (4)} \\ &= f(0) \\ &= 0 \quad \text{(by cty. of } f \text{ at 0).} \end{aligned}$$

Therefore, we have shown given any $x \in \mathbb{R}$, if $\{x_n\}_{n=1}^{\infty}$ is any sequence such that $x_n \to x$ then we have $f(x_n) \to f(x)$. Hence, f is continuous on $\mathbb{R}$. □

5 Question5

5.1 Question 5(a)

We want to show if $f, g \in C(I; \mathbb{R})$ are uniformly continuous and bounded then $fg : I \to \mathbb{R}$ is uniformly continuous on I.

Proof. Since each function is bounded let $M \equiv \sup_{x\in I} |f(x)|$ and $N \equiv \sup_{y\in I} |g(y)|$. Without loss of generality, we can assume that M and N are not identically zero, otherwise fg would be the zero function which is uniformly continuous on I.

Given $\varepsilon > 0$, the uniform continuity of f implies there exists a $\delta_f > 0$ such that:

$$\forall\, x, y \in I: \; |x - y| < \delta_f \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2N} \tag{5}$$

and the uniform continuity of g implies there exists a $\delta_g > 0$ such that

$$\forall\, x, y \in I: \; |x - y| < \delta_g \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{2M}. \tag{6}$$

Consequently, for $\delta < \min\{\delta_f, \delta_g\}$ we have that if $x, y \in I : |x - y| < \delta$ then both (5) and (6) are true, which in turn implies:

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + g(y)f(x) - g(y)f(y)| \\ &\le |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &< M\frac{\varepsilon}{2M} + N\frac{\varepsilon}{2N} = \varepsilon. \end{aligned}$$

This argument is valid for any $\varepsilon > 0$, thus the product fg is uniformly continuous on I. □

5.2 Question 5(b)

The functions $f, g \in C([0\infty);\mathbb{R})$ given below:

$$f(x) \equiv x$$
$$g(x) \equiv -x.$$

are each uniformly continuous on $[0, \infty)$ but the product

$$f(x)g(x) = -x^2$$

is not uniformly continuous on $[0, \infty)$. We give the proof below for reference only, that is, it is not required as part of the solution.

Proof. The proof that f and g are uniformly continuous follows immediately by choosing $\delta = \varepsilon$ in the definition of uniform continuity.

To show that the product is not uniformly continuous, fix $\varepsilon = 1$ and let $\delta > 0$ be given. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be the following sequences:

$$x_n \equiv \sqrt{n+1}$$
$$y_n \equiv \sqrt{n}.$$

Then

$$|x_n - y_n| = |\sqrt{n+1} - \sqrt{n}|\frac{|\sqrt{n+1} + \sqrt{n}|}{|\sqrt{n+1} + \sqrt{n}|} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \quad \forall\ n \in \mathbb{N}.$$

Since $1/\sqrt{n} \to 0$ as $n \to \infty$ there exists an $N \in \mathbb{N}$ such that

$$|x_n - y_n| < \frac{1}{\sqrt{n}} < \delta \quad \forall\ n > N,$$

but

$$|-x_n^2 + y_n^2| = |-n - 1 + n| = 1 \geq \varepsilon.$$

Thus the product is not uniformly continuous on $[0, \infty)$ □