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Therefore,

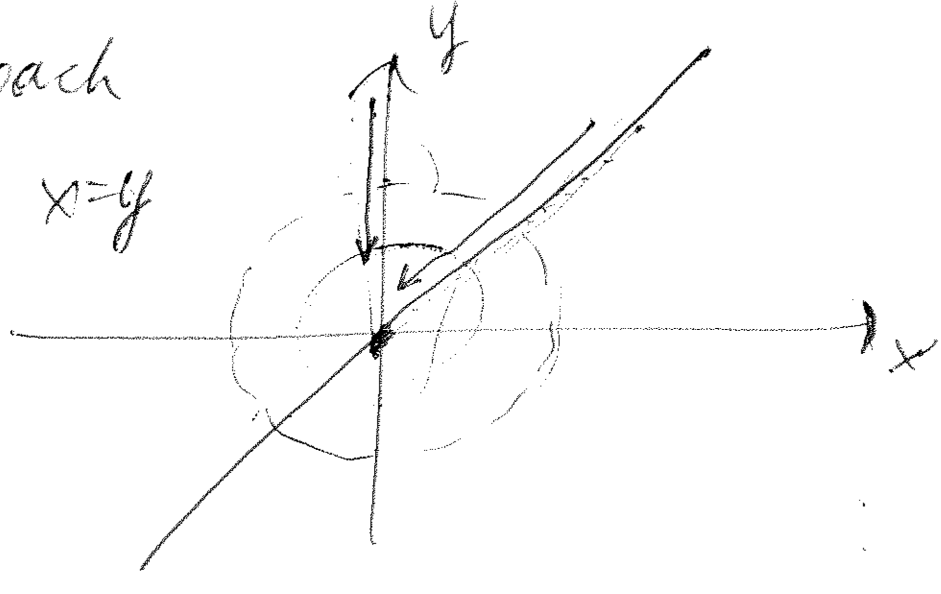
$$|1-L| < \frac{1}{4}, \quad |L| > \frac{3}{4}.$$

We have $|L| < \frac{1}{4}$, $|L| > \frac{3}{4}$, impossible. $\blacksquare$

Hence lim does not exist. $\square$

Intuitively, we are approaching the origin first along the y-axis $(x=0)$.

Then we approach along the line $x=y$



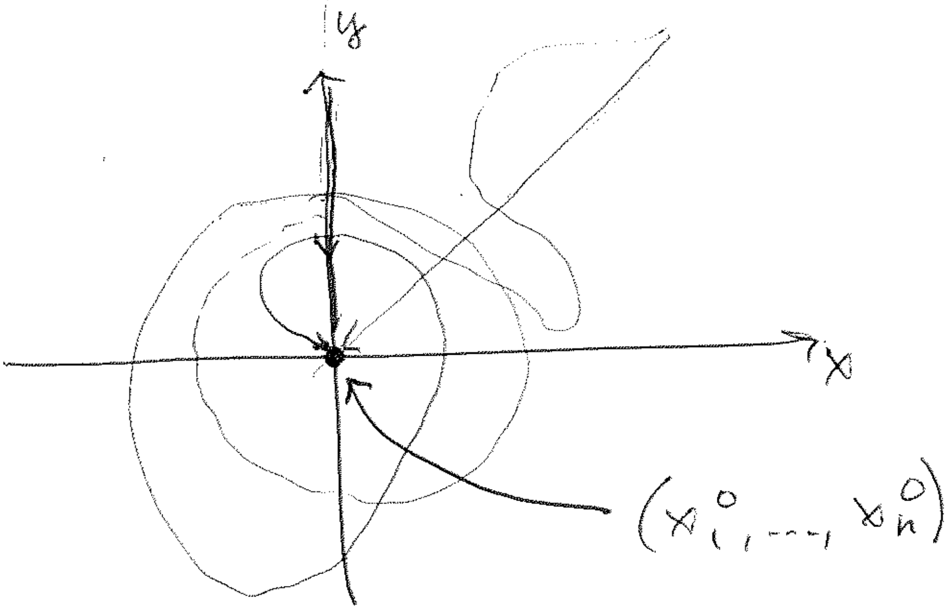
Limits of vector-functions

Consider $f: \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \longrightarrow \mathbb{R}^m$.

We study

$$\lim_{(x_1,\dots,x_n)\to(x_1^0,\dots,x_n^0)} f(x_1,\dots,x_n)$$

<u>Intuition.</u> If this limit exists and equals $L \in \mathbb{R}^m$, that mean that as $(x_1,\dots,x_n)$ approach $(x_1^0,\dots,x_n^0)$, the values $f(x_1,\dots,x_n)$ approach $L \in \mathbb{R}^m$.



However I approach $(x_1^0,\dots,x_n^0)$, if the limit exists, it must be the same.

<u>Ex.</u> The limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+y^2+zx}{x^2+y^2+z^2}.$$

Approach the origin along the z-axis. Then $x, y = 0$ and

$$\frac{xy+y^2+zx}{x^2+y^2+z^2} = 0$$ on the z-axis.

Hence the function approach 0.

But! Let us approach along $x = y = z$. Then along this line,

$$\frac{xy+y^2+zx}{x^2+y^2+z^2} = \frac{x^2+\cancel{y}x^2+x^2}{x^2+x^2+x^2} = 1.$$

The function approaches 1.

Thus, the limit does not exist.

Ex. The limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$

does not exist.

Why? First, approach along $y=0$. Then $\frac{x^2y}{x^4+y^2}$ approaches 0.

Approach along $x=y$. Then

$$\frac{x^2 y}{x^4+y^2} = \frac{x^3}{x^4+x^2}$$

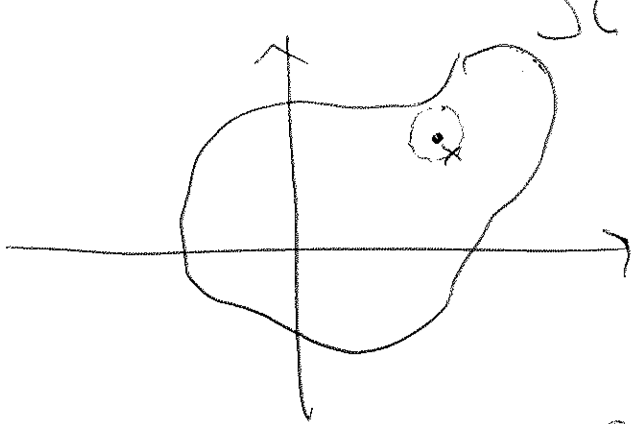
Approach along $y=x^2$. Then

$$\frac{x^2 y}{x^4+y^2} = \frac{x^4}{x^4+x^4} = \frac{1}{2} \neq 0.$$

Thus, lim does not exist.

Derivatives of vector functions.

Def. Suppose $\Omega \subset \mathbb{R}^n$. We say Ω is open if for every $x \in \Omega$, there exists $\delta >$ s.t. if $|y-x| < \delta$, then $y \in \Omega$.



Def. Assume
$$f : \underset{\mathbb{R}^n}{\overset{\cap}{\Omega}} \to \mathbb{R}^m$$
with Ω open. We can write f as $f = (f_1, f_2, \dots, f_m)$ with $f_i : \Omega \to \mathbb{R}$ for all $i = 1, \dots, m$.
The Jacobian matrix of f is

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

where

$\frac{\partial f_j}{\partial x_k}$ is the derivative of f_j with respect to x_k when we consider the other x_i's constants.

<u>Ex.</u> $f(x,y) = (\overbrace{x^2+xy}^{f_1}, \overbrace{x+y^3}^{f_2})$

$$\frac{\partial f_1}{\partial x} = 2x+y, \quad \frac{\partial f_1}{\partial y} = x.$$

$$J_f(x,y) = \begin{pmatrix} 2x+y & x \\ 1 & 3y^2 \end{pmatrix}.$$

<u>Def.</u> The determinant of the Jacobian matrix is called the <u>Jacobian</u> of f.

Here $\det J_f(x,y) = (2x+y)\cdot 3y^2 - x$.

Inverse function theorem.

$f : \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \to \mathbb{R}^m$.

Does f have an inverse?
(the inverse of f is $f^{-1} : f(\Omega) \to \Omega$ such that $f(f^{-1}(x)) = x$ for all $x \in \Omega$).

$f(x) = x^2$,
$f : [0, \infty) \to \mathbb{R}$,
$f^{-1}(x) = \sqrt{x}$.

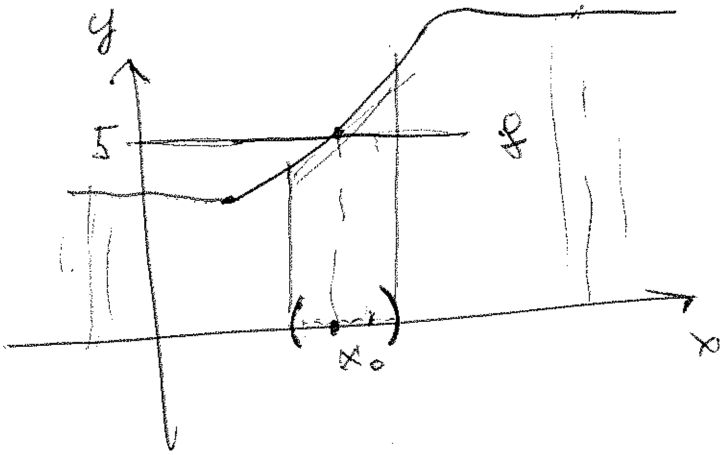
Theorem (Inv. Func. Thm.)
Assume $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ has all $\frac{\partial f_i}{\partial x_j}$ continuous for $i = 1, \dots, m$, $j = 1, \dots, n$.
Assume the Jacobian

$$\det J_f(x_0) \neq 0$$

for some $x_0 \in \mathbb{R}^n$. Then there exists an open set Ω s.t. $x_0 \in \Omega$ and $f : \Omega \to f(\Omega)$ ~~has~~ has a continuous inverse.

Also,

$$J_{f^{-1}}(f(x_0)) = J_f^{-1}(x_0).$$



<u>Ex.</u> $f(x,y) = (x^3 - 2xy^2,\ x+y)$,

$$f : \mathbb{R}^2 \to \mathbb{R}^2.$$

Take $(x_0, y_0) = (1, -1)$.

Is f invertible near (x_0, y_0)?

Can we find an open $\Omega \ni (x_0, y_0)$ s.t. f is invertible in Ω.

$J_f(x,y) =$

$$\begin{pmatrix} 3x^2 - 2y^2 & -4xy \\ 1 & 1 \end{pmatrix}.$$

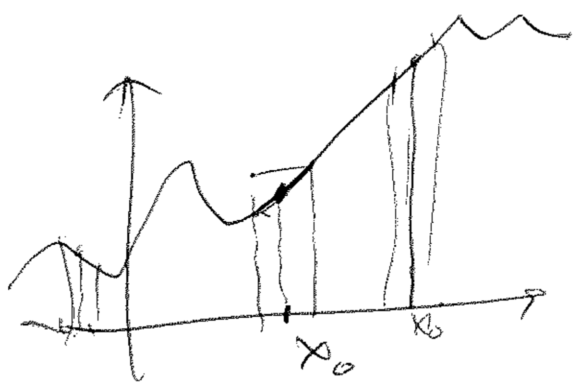
$$J_f(1,-1) = \begin{pmatrix} 3-2 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

$$\det J_f(1,-1) = -3 \neq 0.$$

Thus, f is inverible near $(1,-1)$.

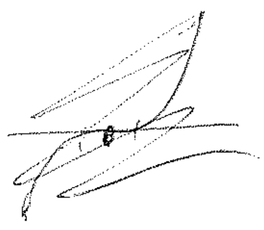
<u>Ex:</u> $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^{26} - y^{26}, xy)$.

Invertible near every $(x,y) \neq 0$.
(locally invertible).



Use Inv. F.T.:

$$J_f(x,y) = \begin{pmatrix} 26x^{25} & -26y^{25} \\ y & x \end{pmatrix},$$



$$\det J_f(x,y) = 26x^{26} + 26y^{26} \neq 0$$

as long as $(x,y) \neq (0,0)$.

Is f globally invertible?
(i.e., does it have an inverse on <u>all</u> of $\mathbb{R}^2$?)

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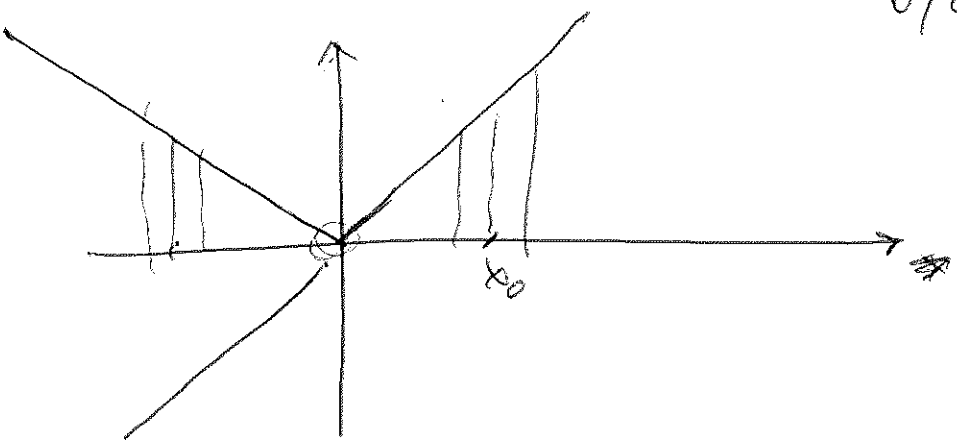
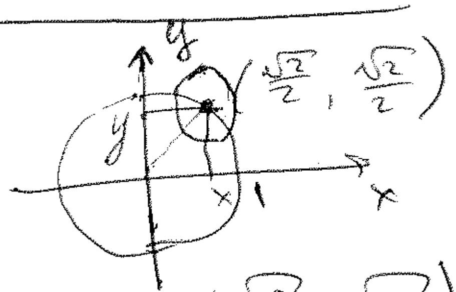


f is NOT globally invertible because

$f(1,1) = f(-1,-1) = (0,1)$.

Implicit func. thm.



Consider $x^2 + y^2 = 1$.

We can solve this for y near $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$y = + \sqrt{1 - x^2}$

Theorem. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set. Let $F : \Omega \to \mathbb{R}^m$ be a function with continuous derivatives. Assume $(\underset{\mathbb{R}^n}{x_0}, \underset{\mathbb{R}^m}{y_0}) \in \Omega$ and $\det J_F(x_0, y_0) \neq 0$. and $F(x_0, y_0) = 0$

Then there are open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ s.t. $x_0 \in U$, $y_0 \in V$, and there is a unique function $f: U \to V$ s.t.

$$F(x, f(x)) = 0, \quad x \in U.$$

Basically, "$y = f(x)$".

Example.

Can we solve

$$\begin{cases} xu + yv^2 = 0, \\ xv^3 + y^2u^6 = 0, \end{cases}$$

for u, v near $x=1, y=-1, u=1, v=-1$?

Take $F = (F_1, F_2): \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$F_1(x, y, u, v) = xu + yv^2,$$

$$F_2(x, y, u, v) = xv^3 + y^2u^6.$$

$$F(1, -1, 1, -1) = 0,$$

$$J_F(x, y, u, v) = \begin{pmatrix} x & 2vy \\ 6u^5y^2 & 3xv^2 \end{pmatrix}.$$

$$J_F(1, -1, 1, -1) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}, \quad \det J_F(1, -1, 1, -1) = -9 \neq 0.$$

Now, the implicit func. theorem tells us that we can solve for u, v $\left(u = f_1(x,y),\ v = f_2(x,y) \right)$

in some neighbourhood of $x=1, y=-1, u=1, v=-1$.

Here, $\xi \in (0,x)$ or $(x,0)$.

Take $x=1$. Then

$$e = 1+1+\frac{1}{2}+\cdots+\frac{1}{n!}+\frac{e^{\xi}}{(n+1)!}\cdot 1^{n+1},$$

$\xi \in (0,1)$. Thus, $e^{\xi} \in (1,e)$.

We know $e \leq 3$. Therefore,

$e^{\xi} < 3$.

This means the error term

$$\frac{e^{\xi}}{(n+1)!} < \frac{3}{(n+1)!}.$$

We want this to be

< 0.000005.

Take $n=9$, this will be enough.

Thus,

$$e = 1+1+\frac{1}{2}+\cdots+\frac{1}{9!}+R_9(e), \text{ where}$$

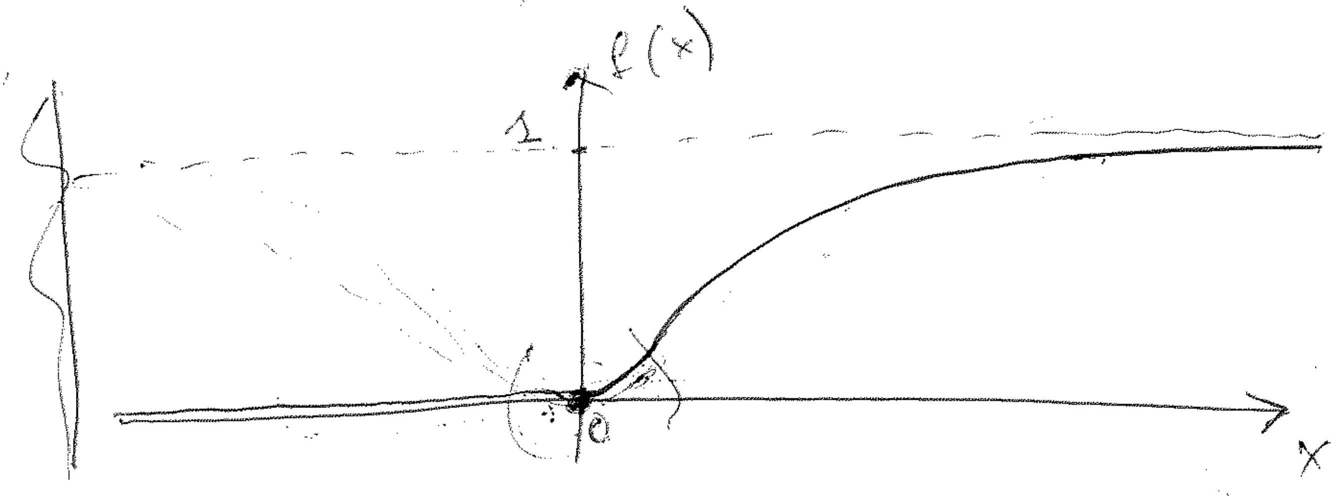
$|R_9(e)| < 0.000005$. Also, we know $R_9(e) > 0$. Thus,

$e \in \left(1+1+\cdots+\frac{1}{9!},\ 1+1+\cdots+\frac{1}{9!}+0.000005\right)$.

That does it.

<u>Example.</u>

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x>0, \\ 0, & x \le 0. \end{cases}$$



Clearly, f is diff when $x \neq 0$.

Also, we can verify that f is infinitely differentiable at 0.

We can use Taylor's theorem to write f as a series about $x=0$.

$$f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \text{ when } x>0.$$

But! $f^{(n)}(0) = 0$ for all n. Thus, the series is not a "good" representation for f.

Note. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

For this, $P_0 = P_1 = 1$, $P_2 = P_3 = 1 - \frac{x^2}{2!}$, ...

Therefore, $R_0 = R_1$, $R_2 = R_3$, ...

Vector functions.

Notation. Suppose $x, y \in \mathbb{R}^m$,

$$x = (x_1, \dots, x_m), \quad y = (y_1, \dots, y_m).$$

$$|x| = \|x\| = \left[\sum_{i=1}^{m} x_i^2\right]^{\frac{1}{2}}.$$

$$(x, y) = \sum_{i=1}^{m} x_i y_i.$$

Theorem. If $x, y \in \mathbb{R}^m$, then

1) $|(x, y)| \le |x||y|$

2) $|x + y| \le |x| + |y|$.

Proof. 1) If $x, y = 0$, then this is obvious.

Assume $x \neq 0$, $y \neq 0$.

Observation: If $a, b \in \mathbb{R}$, then

$$\underline{a^2 - 2|a||b| + b^2 \geq 0} \Rightarrow |ab| \leq \frac{1}{2}(a^2 + b^2).$$

Now take $a = \frac{|x_i|}{|x|}$ and $b = \frac{|y_i|}{|y|}$

for some $i = 1, \ldots, m$. Then

$$\frac{|x_i y_i|}{|x||y|} \leq \frac{1}{2}\left(\frac{|x_i|^2}{|x|^2} + \frac{|y_i|^2}{|y|^2}\right).$$

Now sum these ineqs. for $i = 1, \ldots, m$.

Obtain:

$$\underline{\sum_{i=1}^{m} \frac{|x_i y_i|}{|x||y|}} \leq \frac{1}{2} \sum_{i=1}^{m} \left(\frac{|x_i|^2}{|x|^2}\right) + \frac{1}{2} \sum \frac{|y_i|^2}{|y|^2}$$

$$= \frac{1}{2} \frac{1}{|x|^2} \underbrace{\left(\sum_{i=1}^{m} |x_i|^2\right)}_{=|x|^2} + \frac{1}{2} \frac{1}{|y|^2} \underbrace{\left(\sum_{i=1}^{m} |y_i|^2\right)}_{=|y|^2}$$

$$= \frac{1}{2} + \frac{1}{2} = \underline{\underline{1}}.$$

Thus, $\sum_{i=1}^{m} |x_i y_i| \leq \underline{|x| \cdot |y|}$. Also,

by the triangle ineq. in $\mathbb{R}$, $\underline{\left|\sum_{i=1}^{m} x_i y_i\right|} \leq \underline{\sum_{i=1}^{m} |x_i y_i|}$.

Thus, $\underline{|x||y|} \geq \left|\sum_{i=1}^{n} x_i y_i\right| = \underline{|(x,y)|}$.

2) Note that $|x+y|^2 = (x+y, x+y)$

$= (x, x) + (x,y) + (y,x) + (y,y)$

$= |x|^2 + 2(x,y) + |y|^2$

$\leq |x|^2 + 2|(x,y)| + |y|^2$

$\leq |x|^2 + 2|x||y| + |y|^2$

$= (|x|+|y|)^2$. Thus,

$|x+y|^2 \leq (|x|+|y|)^2$, take $\sqrt{}$.

$\square$

<u>Notes</u> 1) $|x| = 0 \Leftrightarrow x = 0$.

2) $(x, 0) = 0$ for all x.

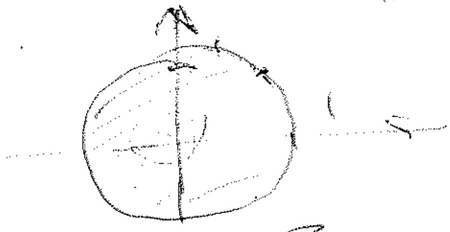
3) $(x,y) = 0 \not\Rightarrow x = 0$ or $y = 0$.

Vector functions

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\quad f: \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \longrightarrow \mathbb{R}^m$.

Example. $\quad f(x_1, x_2) = \left(\sqrt{1-x_1^2-x_2^2}\,,\ x_1+x_2\right)$.

$n = m = 2$.

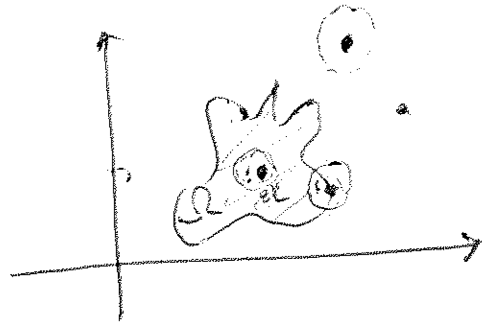


Domain of f is

$\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 \leq 1\}$.

$f: \Omega \longrightarrow \mathbb{R}^2$.

Def. Suppose Ω is a subset of $\mathbb{R}^n$. The point a is a limit point of Ω if for every $\varepsilon > 0$ there exists $y \in \Omega$ s.t. $0 < |y - a| < \varepsilon$.



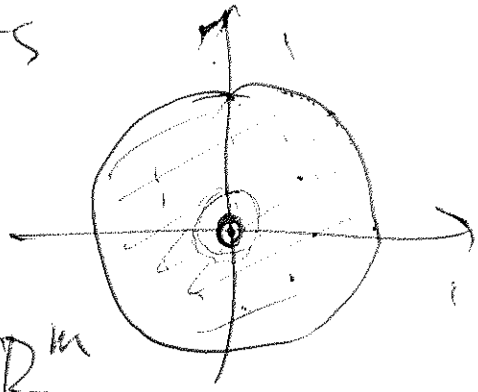
Ex. 1) If $\Omega = B_1(0)$

$= \{x \in \mathbb{R}^n \mid |x| < 1\}$,

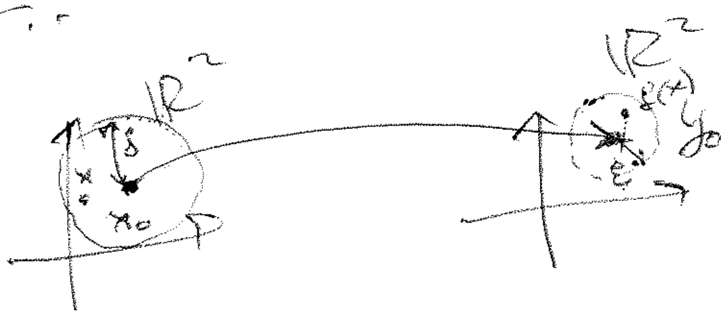
then the set of limit points of Ω is

$\{x \in \mathbb{R} \mid |x| \leq 1\}$.

2) $\Omega = B_1(0) \setminus \{(0,0)\}$. Then the set of limit points is the same:



<u>Def.</u> Let $f : \Omega \to \mathbb{R}^m$ be a function (here, $\Omega \subset \mathbb{R}^n$). Let x_0 be a limit point of Ω and $y_0 \in \mathbb{R}^m$. Then $\lim_{x \to x_0} f(x) = y_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $\underline{0 < |x - x_0| < \delta}$, and $x \in \Omega$, then $|f(x) - y_0| < \varepsilon$.



<u>Def.</u> f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

<u>Remark.</u> Every $f : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $(f_1, f_2, f_3, \dots, f_m)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \dots, m$.

<u>Fact.</u> $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f_1, \dots, f_m$ are all continuous.

<u>Example.</u> $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$,

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Question: does $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ exist?

If so, what is it?

I claim the limit exists.

Let's prove it.

Fix $\varepsilon > 0$. We want to find $\delta > 0$ s.t. if $0 < |(x,y)-(0,0)| < \delta$, then

$\left| \frac{xy}{\sqrt{x^2+y^2}} - L \right| < \varepsilon$ for some L.

Now, $|(x,y)-(0,0)| = \sqrt{x^2+y^2} < \delta$.

Note that

$$(|x|-|y|)^2 \geq 0 \Rightarrow x^2+y^2-2|xy| \geq 0$$
$$\Rightarrow \underline{x^2+y^2 \geq 2|xy|}.$$

Now,

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\sqrt{x^2+y^2} < \frac{\delta}{2}$$

Thus, if we set $\delta = 2\varepsilon$, then

$|(x,y)-(0,0)| < \delta$ will imply

$$|f(x,y)-0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \varepsilon.$$

Thus, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

2) $f(x,y) = \dfrac{xy^2}{x^2+y^2}$, find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Claim: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0$. Why?

Note that $|f(x,y)| = \dfrac{|x|\,y^2}{x^2+y^2} \le |x|$.

If $0 < |(x,y)-(0,0)| < \delta$, then $(x,y) \neq 0$.

$$|f(x,y)-0| = |f(x,y)| \le |x| \le \sqrt{x^2+y^2} < \delta.$$

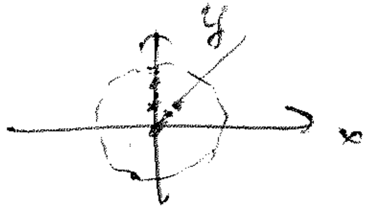
Now take $\varepsilon = \delta$. If $|(x,y) - (0,0)| < \delta$, then

$|f(x,y)| < \delta = \varepsilon$.

Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

<u>Example.</u> Does $\lim_{(x,y)\to(0,0)} \dfrac{x^2+xy}{x^2+y^2}$ exist?

If it existed, for all (x,y) s.t. $0 < |(x,y)-(0,0)| < \delta$, we would have $\left|\dfrac{x^2+xy}{x^2+y^2} - L\right| < \varepsilon$.



Take $\varepsilon = \frac{1}{4}$. Take $x=0$ and $y \neq 0$. Then

$\left|\dfrac{x^2+x\cdot y}{x^2+y^2} - L\right| = |0-L| < \frac{1}{4}$, $|L| < \frac{1}{4}$.

However, take x, y s.t. $(x=y)$.

No matter how close (x,y) is to $(0,0)$, we then have $\left|\dfrac{x^2+xy}{x^2+y^2} - L\right| = \left|\dfrac{2x^2}{2x^2} - L\right| = |1-L| < \frac{1}{4}$.