Math 2400
Assignment 2
Due 11:50 a.m. on 9 April, 2014

Drop your paper into the white box with your tutorial number on the fourth floor of the Priestley building

1. (10 points) Prove that the set of triples $\{(p, q, r) \mid p, q, r \in \mathbb{N}\}$ is countable.

2. (10 points) For every $x \in \mathbb{R}$, compute

$$\lim_{n\to\infty} \left(\lim_{k\to\infty} (\cos n!\pi x)^{2k} \right).$$

Here, we use the standard notation $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2) \cdot (n-1) \cdot n$.

3. (10 points) Suppose $(a_n)_{n=1}^{\infty}$ is a convergent sequence and $a_n \in [0, 1]$ for all n. Prove that the limit of $(a_n)_{n=1}^{\infty}$ lies in $[0, 1]$.

4. (10 points) Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers.

   (a) Prove that
   $$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

   (b) Find $\limsup_{n\to\infty}(-1)^n\left(1 + \frac{1}{n}\right)$ and $\liminf_{n\to\infty}(-1)^n\left(1 + \frac{1}{n}\right)$.

5. (10 points) Consider a sequence $(b_n)_{n=1}^{\infty}$. We say that the infinite product $\Pi_{n=1}^{\infty} b_n$ converges to $p \in \mathbb{R} \setminus \{0\}$ if $\lim_{n\to\infty} \Pi_{k=1}^{n} b_k = p$. If the product does not converge to any $p \in \mathbb{R} \setminus \{0\}$, we say it diverges.

   (a) Prove that $\lim_{n\to\infty} b_n = 1$ if $\Pi_{n=1}^{\infty} b_n$ converges.

   (b) Find $\Pi_{n=1}^{\infty} \frac{n^3+n^2+n}{n^3+1}$ or show that the product diverges.