\[ (i) \lim_{n \to \infty} x^{2n} = 0 \quad (0 \leq x < 1) \]
\[ \lim_{n \to \infty} n^2 = 1 \]
\[ \implies f = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases} \]

on a closed, bounded interval.

(ii) No. If it were, the limit \( f^* \) would have to be continuous (uniform limit of continuous functions is continuous), which it isn't.

(iii) \( \log x \in C_{[0,1]} \) & \( f_n \to f \) on \( [0,1] \), so \( f_n \to f \) on \( [0,1] \).

(iv) See last page.

2) \( f \in C_{[0,1]} \). By Rolle's theorem, if \( f \) has 2 distinct roots \( a \) & \( b \) in \( [0,1] \) with \( 0 < a < b < 1 \), then \( \exists c \in (a,b) \) (so \( c \in (0,1) \)) with \( f'(c) = 0 \) (\( f(b) = f(a) \)). But \( f'(c) = 3x^2 - 3 = 3(x^2 - 1) \) is strictly negative on \((0,1)\), which is a contradiction.

3) \( \frac{-1}{n^2} \leq \frac{(\sin x)^n}{n^2} \leq \frac{1}{n^2} \)

So \( 0 \leq \frac{1}{n^2} (\sin x)^n \leq \frac{1}{n^2} \). Since \( \sum \frac{1}{n^2} \) converges \( (p - \text{series}, p > 1) \), \( \frac{1}{n^2} (\sin x)^n \) converges, absolutely meaning that \( \sum \frac{(\sin x)^n}{n^2} \) converges absolutely.

(iii) \( f(x) = \frac{1}{\log x} \) is monotone decreasing & positive on \( [3, \infty) \), with \( \lim_{x \to \infty} \frac{1}{\log x} = 0 \). So by the integral test, \( \sum \frac{1}{\log x} \) converges \( \iff \int_3^\infty \frac{dx}{\log x} \) converges. Put \( y = \log x \), \( dy = \frac{1}{x} dx \)

\[ \int_3^\infty \frac{dy}{y} = -\log y \bigg|_3^\infty = -\infty \] So series diverges. (only positive terms, so it can't be conditionally convergent.)
(iii) Analogous to (ii), as $n \to \infty$, $\frac{1}{\sqrt{n}}$ is monotone decreasing to 0 (as $n \to \infty$).

\[ \int \frac{dy}{y\sqrt{y}} = \log y + \sqrt{y} + C \]

which converges, so the series converges absolutely.

(iv) $0 < \frac{1}{n^2 \log n} < \frac{1}{n^2}$ for $n \geq 3$.

So, $\sum \frac{1}{n^2 \log n}$ converges absolutely since $\sum \frac{1}{n^2}$.

by the comparison test.

5. (ii) $f^{(n)}(x) = \sinh n \pi x$ even $\Rightarrow f^{(n)}(0) = 0$ even

$cosh x$ n odd $\Rightarrow 1 \text{ n odd}$

$\Rightarrow$ MacLaurin write in $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

$= \sum \frac{x^{2n+1}}{(2n+1)!}$

$x + \frac{x^3}{3!} = \phi_3 = \phi_4$, so the error can be written as $f^{(5)}(\theta)$ for some $\theta \in (0, x)$, so such

\[ \text{for some } \theta \in (0, x), \text{ the error is given by } 5 \frac{\cosh \frac{\theta}{5}}{5!} \leq \theta \frac{25 \cdot 5!}{25!} \approx 0.00029. \]

5. (iii) For $f(x, y, z) = \frac{\beta y + x^2}{\beta y + y^2}$, $A(x, y, z) = (x, \hat{\beta}, 0)$ with $\beta > 0$ we have $f(x, \hat{\beta}, 0) = \frac{\beta}{\beta y + y^2} \Rightarrow \lim_{(x, \hat{\beta}, 0) \to 0} f(x, \hat{\beta}, 0)$ depends on $\hat{\beta}$ (e.g., $x = \hat{\beta} = 1$, limit = $\frac{\beta}{\beta}$; $x = \hat{\beta} = 0$, limit = 0), so $(x, \hat{\beta}, 0)$ doesn't exist.
5) Put \( F(x, y) = \frac{x^4 - y^4}{x - y} \). Since \( F(x, y) \) is undefined for \( x = y \), \( F \) is not defined on any deleted neighborhood \( \mathbb{R} \setminus \{0, 2\} \), so the limit is not defined, according to our definition. However, full marks were given for noting that for \( x \neq y \), \( F(x, y) = (x+y)(x^3 + x^2y + xy^2 + y^3) \)

\[
\lim_{{(x, y) \to (0, 0)}} F(x, y) = \lim_{{(x, y) \to (0, 2)}} (8 + 8 + 8 + 8) = 32
\]

6) (i) \( J_F = \begin{pmatrix} 6x^5 & -6y^5 \\ y & x \end{pmatrix} \) \( \text{det } J_F = 6(x^6 + y^6) \)

(ii) \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is \( C^1 \), with \( \text{det } J_F \neq 0 \) for \((x, y) \neq (0, 0)\).

(iii) \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) has a local inverse near any \((x, y) \neq (0, 0)\).

(iv) \( F(1, 0) = (1, 0) \neq (-1, 0) \Rightarrow F \) is not globally invertible.

4) In order to show uniform convergence:

Given \( \varepsilon > 0 \), need \( N \in \mathbb{N} \) s.t.

\[ m > N \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in E_0, \rho, \]

i.e. \( n > N \Rightarrow 2^n < \varepsilon \quad \forall x \in E_0, \rho \). 

So choose \( N \) s.t. \( 2^N < \varepsilon \) (possibly, since \( x \mapsto 2^x \) is a decreasing function) \( N \). For \( n > N \), there holds: \( 0 \leq \frac{2^n}{\rho} < \frac{2^n}{\rho^2} < \varepsilon \Rightarrow \) \( \star \).